Localised equivariant cohomology and Milnor's additivity

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To all victims of the recent wars in South Slavonic countries.

Lux perpetua luceat eis.

Contents

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Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Summary

Borel equivariant cohomology, $H^*_{\mathbb{T}}(X; R)$, for spaces with circle action is a module over $R[u]$, where u has degree 2. Localised equivariant cohomology, $u^{-1}H^*_{\mathbb{T}}(X; R)$, provides certain information about $H^*_{\mathbb{T}}(X; R)$, but as a cohomology theory it has a deficiency in that it does not satisfy Milnor's additivity axiom.

In this thesis an equivariant cohomology theory, $h^*_{\mathbb{T}}$, is constructed which agrees with $u^{-1}H^*_{\mathbb{T}}$ on nice finite-dimensional spaces with circle action and satisfies Milnor's additivity axiom. This has already been done by J.D.S. Jones and S.B. Petrack for the smooth case, where a different method was used which cannot be extended to the general case. We show that $h^*_{\mathbb{T}}$ satisfies a fixed point theorem similar to the one for $u^{-1}H^*_{\mathbb{T}}$ on finite dimensional spaces.

Finally, cellular versions of $H^*_{\mathbb{T}}(X; \mathbb{Z})$ and $u^{-1}H^*_{\mathbb{T}}(X; \mathbb{Z})$ are constructed.

Introduction

The purpose of this thesis is to construct a suitable extension of localised equivariant cohomology as defined on nice compact spaces with circle action to infinite dimensional spaces with circle action. The maps into a point make the Borel equivariant cohomology $H^*_{\mathbf{T}}(-;R)$ into a module over the graded ring $R[u]$, where R is a ring and u has degree two. We can formally invert the multiplicative system $\{1, u, u^2, \dots\}$ and obtain localised equivariant cohomology $u^{-1}H^*_{\mathbf{T}}(-;R)$. Localised equivariant cohomology of a $\mathbf{T}$-space X gives some information about the equivariant cohomology, but as a cohomology theory on infinite dimensional spaces it has a serious disadvantage in that it does not satisfy Milnor's additivity axiom. This axiom requires that the cohomology of any disjoint union be the direct product of the cohomologies of the components. Cohomology theories satisfying this axiom have strong uniqueness properties on infinite dimensional spaces [Milnor 62].

The first result in this direction is [Jones, Petrack 88]. There a de Rham equivariant cohomology theory $h^*_{\mathbf{T}}(-;\mathbf{C})$ is constructed, which satisfies Milnor's axiom and agrees with $u^{-1}H^*_{\mathbf{T}}(-;\mathbf{C})$ on finite dimensional smooth $\mathbf{T}$-manifolds. Jones and Petrack were motivated by the cohomological investigation of the free loop space of a manifold with the circle action being the rotation of the loops. On finite dimensional $\mathbf{T}$-spaces the inclusion of the fixed points induces an isomorphism in $u^{-1}H^*_{\mathbf{T}}(-;\mathbf{F})$, if $\mathbf{F}$ is a field of characteristic zero. In [Goodwillie 85] it is shown, however, that this is not true for the free loop spaces. The equivariant cohomology of [Jones, Petrack 88] does satisfy the fixed point theorem also on nice infinite dimensional $\mathbf{T}$-manifolds and in particular on the free loop spaces. Jones and Petrack have also shown that this theory has several interesting properties on the free loop spaces [Jones, Petrack 90].

In this work we construct an appropriate singular version of this theory. More precisely we construct an equivariant cohomology theory $h^*_{\mathbf{T}}$ which we call the Jones-Petrack cohomology, defined for any coefficients, which satisfies Milnor's additivity axiom and agrees with $u^{-1}H^*_{\mathbf{T}}$ on all compact $\mathbf{T}$-spaces with finite cohomological dimension. It satisfies the fixed point theorem if the order of each isotropy subgroup of non-fixed points is invertible in the coefficient ring. It turns out that this generalisation is not quite straightforward and that different methods have to be used to obtain it.

In Chapter 1 we define the Borel equivariant cohomology, $H^*_{\mathbb{T}}(X)$, of a space with a circle action. We present a basic fixed point theorem for localised equivariant cohomology,$u^{-1}H^*_{\mathbb{T}}$, and show that it does not satisfy Milnor's additivity axiom. We describe also the equivariant cohomology of [Jones, Petrack 88].

In Chapter 2 we show that an obvious singular version of the approach of Jones and Petrack does not have the required properties. Their approach is in some sense global and is good for the de Rham version of equivariant cohomology. The approach we use to obtain the desired cohomology theory is local. The crucial property of $\mathbb{T}$-spaces which we use is the slice theorem in its topological and differential form. The slice theorem implies that infinite dimensional $\mathbb{T}$-spaces also have locally nice properties.

Therefore our strategy is the following: we calculate localised equivariant cohomology locally and re-assemble this local information in a suitable way to obtain the desired cohomology theory. To do this we use differential graded sheaves.

In Chapter 3 we prove that localised equivariant cohomology on compact admissible $\mathbb{T}$-pairs is isomorphic to Čech cohomology of differential graded presheaves $S^*(-)[u, u^{-1}]$. These DG presheaves are $\mathbb{Z}$-graded and unbounded in both the positive and negative directions. Therefore the standard DG (pre)sheaf cohomology does not have the properties necessary to obtain an equivariant cohomology with Milnor's additivity property. Assuming that we have a suitable DG sheaf cohomology we present the desired Jones-Petrack cohomology.

In Chapter 4 we construct a modified version of the DG sheaf cohomology which gives the desired theory when applied to the DG sheaves obtained by the double complexes $S^*(-)[u, u^{-1}]$.

In Chapter 5 we show that Jones-Petrack cohomology agrees with localised equivariant cohomology on all finite dimensional $\mathbb{T}$-complexes. We present cellular versions of equivariant cohomology and localised equivariant cohomology with integer coefficients.

Appendix A is auxiliary. We need it to prove that Čech cohomology of DG presheaves is isomorphic to the canonical resolution cohomology of the associated DG sheaves.

Chapter 1

Equivariant cohomology

This is a purely introductory chapter and it contains no original results. The references are [Bredon 72], [Jones 87], [Hsiang 75], [Atiyah,Bott 84].

1.1. Borel equivariant cohomology

Let G be a topological group and let X be a left G-space. Then $H_G^*(X)$ the Borel equivariant cohomology, is defined as follows.

Let $E_G \to B_G$ be a fixed universal G-bundle, i.e. E_G is a contractible space with a free, locally trivial G-action and $E_G \to B_G$ is the quotient map onto the orbit space. For a G-space X we define the homotopy quotient

$$X_G = E_G \times_G X = (E_G \times X)/G$$

where G acts diagonally on $E_G \times X$. The equivariant cohomology of X is defined to be the ordinary cohomology of X_G:

$$H_G^*(X) = H^*(X_G).$$

If A is an invariant subspace of X we define the relative equivariant cohomology $H_G^*(X, A)$ by

$$H_G^*(X, A) = H^*(X_G, A_G).$$

The constant map $X \to \mathrm{pt}$ gives a homomorphism of equivariant cohomology $H_G^*(\mathrm{pt}) \to H_G^*(X)$ and the cup product makes $H_G^*(X)$ and $H_G^*(X, A)$ modules over $H_G^*(\mathrm{pt})$.

For homogeneous G-spaces, G/H, for H a closed subgroup of G, we have the following identity

$$(G/H)_G = E_G \times_G (G/H) = (E_G \times_G G)/H = E_G/H = B_H$$

and G-equivariant maps between homogeneous G-spaces are given by the quotient homomorphisms

$$G/H \longrightarrow G/K$$

for pairs of closed subgroups $H \subseteq K$.

We shall only consider the group $\mathbf{T}$ of complex numbers of unit norm and its closed subgroups, which are all finite cyclic groups. We shall need the following information about their cohomology with coefficients in a ring R [Borel 60].

$$H^*(B_{\mathbf{T}}; R) = R[u]\,, \quad n.H^i(B_{\mathbf{Z}/n}; R) = 0 \quad \text{for } i > 0$$

where u has degree 2. In particular

$$H^*_{\mathbf{T}}(\mathbf{T}/(\mathbf{Z}/n); \mathbf{Z}) = H^*(B_{\mathbf{Z}/n}; \mathbf{Z}) = \begin{cases} \mathbf{Z} & * = 0 \\ (\mathbf{Z}/n)u^m & * = 2m \\ 0 & * \text{ odd.} \end{cases}$$

1.2. A double complex for equivariant cohomology

There is another model for $\mathbf{T}$-equivariant cohomology:

Let X be a space with a circle action $\Phi : \mathbf{T} \times X \to X$. Then we define an operator

$$\mathrm{I} : S^*(X) \to S^{*-1}(X)$$

as follows. Let $\mathrm{sh} : S_*(\mathbf{T}) \otimes S_*(X) \to S_*(\mathbf{T} \times X)$ denote the shuffle product (its definition is stated in Chapter 2). Define the slant product α/σ by

$$< \alpha/\sigma, \tau > = < \alpha, \mathrm{sh}(\sigma \otimes \tau) >$$

for $\alpha \in S^*(\mathbf{T} \times X), \sigma \in S_*(\mathbf{T}), \tau \in S_*(X)$. Let v be the fundamental 1-simplex in $\mathbf{T}$ given by $t \mapsto e^{2\pi i t} \in \mathbf{T}$. Then we define

$$\mathrm{I}(\psi) = \Phi^*(\psi)/v$$

for a cochain $\psi \in S^*(X)$.

Then, as we shall show in Propositions 2.2.2 and 2.2.3,

$$\mathrm{I}^2 = 0\,, \quad \delta\mathrm{I} + \mathrm{I}\delta = 0$$

where $\delta : S^*(X) \to S^{*+1}(X)$ is the coboundary operator.

Let u be an indeterminate of degree 2 and define $S^*(X)[u]$ to be the graded module

$$S^*(X; \mathbf{Z}) \otimes \mathbf{Z}[u]\,.$$

The operator $\delta_{\mathbf{T}} = \delta + \mathrm{I}u : S^*(X)[u] \to S^*(X)[u]$ is a differential. Then it is proved in [Jones 87, §5] that

$$\mathrm{H}(S^*(X; \mathbf{Z}) \otimes \mathbf{Z}[u]; \delta_{\mathbf{T}}) = H^*_{\mathbf{T}}(X)$$

as $\mathbf{Z}[u]$-modules.

1.3. The smooth version

For compact connected Lie groups there is a de Rham version of the equivariant cohomology based on Weil's de Rham model for the universal bundle $E_G \to B_G$. Here, following [Atiyah,Bott 84, §4], we outline this model in the case $G = \mathbf{T}$.

Let M be a smooth manifold with a smooth circle action, and let Ω^*_{inv} denote the invariant differential forms with complex coefficients. The circle action determines a smooth vector field V on M (by a choice of a basis vector in the Lie algebra of the circle) and let us denote by

$$\iota_V : \Omega^*_{inv} \to \Omega^{*-1}_{inv}$$

the interior product of differential forms with this vector field. Because of the fundamental homotopy identity

$$\mathfrak{L}_V = \iota_V d + d\,\iota_V$$

where $\mathfrak{L}_V$ is the Lie derivative, the operator ι_V anticommutes with the exterior derivative d on the invariant forms Ω^*_{inv}. As above we construct a cochain complex

$$\Omega^*_{inv}(M)[u] = \Omega^*_{inv}(M) \otimes_{\mathbf{C}} \mathbf{C}[u]$$

where u is an indeterminate of degree 2, and the differential is $d_{\mathbf{T}} = d + \iota_V u$. In [Atiyah,Bott 84] it is proved that the homology of this cochain complex is isomorphic to the equivariant cohomology of M as $\mathbf{C}[u]$-modules.

1.4. A fixed point theorem

Here we give a simple proof of A. Borel's result about the rational $\mathbf{T}$-equivariant cohomology [Borel 60], [Atiyah,Bott 84], [Hsiang 75].

Let us first recollect a fundamental theorem about the local properties of $\mathbf{T}$-spaces. It is called the slice theorem and it is used in the proof of the fixed point theorem. Note that it holds also for infinite dimensional $\mathbf{T}$-spaces.

Theorem 1.4.1. *Let G be a compact Lie group and X a completely regular topological space with a (continuous) G-action. Let $x \in X$, then there exists a subset S in X such that*

(a) S is invariant under the isotropy group G_x of x,

(b) Let $G(S)$ be the minimal G-invariant subspace containing S. Then

$$G(S) \cong_G G \times_{G_x} S$$

is an invariant open neighbourhood of $G(x)$ in X.

The subset S is called a slice at x. The proof of this theorem can be found in [Bredon 72]. The differentiable slice theorem is as follows.

Theorem 1.4.2. *Let M be a smooth manifold with a smooth G-action and an invariant metric. Let $H = G_x$ be the isotropy subgroup at a point $x \in M$, and ϕ_x be the local representation of H on normal vectors of the orbit $G(x) \cong G/H$ at x. Then, the equivariant normal bundle $\nu(G(x))$ of $G(x)$ is isomorphic to*

$$G \times_H \mathbb{R}^k \longrightarrow G/H$$

where H acts on G as right translations and on $\mathbb{R}^k$ via ϕ_x. Furthermore, for a sufficiently small $\epsilon > 0$, the exponential map is an equivariant diffeomorphism of the associated ϵ-disc bundle of $G \times_H D^k \to G/H$ onto an invariant tubular neighbourhood of $G(x)$.

The proof of this theorem can be found in [Hsiang 75].

Let R be a ring with unit. Recall that a multiplicative semigroup S which contains the unit $1 \in R$ and is contained in the centre of R is called a *multiplicative system*. The localised module $S^{-1}M$ of an R-module M consists of all fractions $\{m/s; m \in M, s \in S\}$ with the usual identification $m_1/s_1 = m_2/s_2$ if and only if $ss_1m_2 = ss_2m_1$ for some $s \in S$. Then $S^{-1}M$ is an $(S^{-1}R)$-module and $M \mapsto S^{-1}M$ is an exact functor.

We can localise $\mathbf{T}$-equivariant cohomology with respect to the multiplicative system $\{u^n; n \geq 0\} \subset \mathbb{Z}[u]$. The resulting equivariant cohomology theory will be denoted by $u^{-1}H^*_{\mathbf{T}}$ and referred to as localised equivariant cohomology.

For localised equivariant cohomology the following fixed point theorem holds.

Theorem 1.4.3. *Let X be a compact $\mathbf{T}$-space and F the fixed point subspace, then the inclusion $F \hookrightarrow X$ induces an isomorphism*

$$u^{-1}H^*_{\mathbf{T}}(X; \mathbf{Q}) \cong u^{-1}H^*_{\mathbf{T}}(F; \mathbf{Q}).$$

Proof of 1.4.3: First let us prove the theorem for the case $F = \emptyset$. By the slice theorem every orbit $\mathbf{T}.x$ has an invariant open neighbourhood U which retracts by an equivariant retraction r onto the orbit. Therefore the homomorphism

$$r^* : H^*_{\mathbf{T}}(\mathbf{T}.x; \mathbf{Q}) \to H^*_{\mathbf{T}}(U; \mathbf{Q})$$

is injective. As x is not a fixed point $H^*_{\mathbf{T}}(\mathbf{T}.x; \mathbf{Q}) = 0$ for $* \neq 0$. Therefore the homomorphism $H^*_{\mathbf{T}}(\mathrm{pt}; \mathbf{Q}) = \mathbf{Q}[u] \to H^*_{\mathbf{T}}(\mathbf{T}.x; \mathbf{Q})$, induced by the constant map, maps u into 0. The same must then be true for $H^*_{\mathbf{T}}(\mathrm{pt}; \mathbf{Q}) \to H^*_{\mathbf{T}}(U; \mathbf{Q})$. Because X is compact, finitely many sets $U_1, \dots, U_q$, such that each of them equivariantly retracts onto an orbit, cover X. From the exact sequence of the pair it follows that there is an element $u_i \in H^2_{\mathbb{T}}(X, U_i; \mathbb{Q})$ which is mapped by the restriction homomorphism into $u \in H^2_{\mathbb{T}}(X)$. Therefore $u^q \in H^{2q}_{\mathbb{T}}(X)$ is the image of

$$u_1 \cdots u_q \in H^{2q}_{\mathbb{T}}(X, U_1 \cup \cdots \cup U_q; \mathbb{Q}) = 0$$

and so $u^q = 0$.

For the general case let us show that $H^*_{\mathbb{T}}(X, F; \mathbb{Q})$ is a torsion $\mathbb{Q}[u]$-module. Let $\phi \in H^n_{\mathbb{T}}(X, F; \mathbb{Q})$ and let $p : X_{\mathbb{T}} \to B_{\mathbb{T}}$ denote the map induced by the constant map $X \to \mathrm{pt}$. We may assume that $B_{\mathbb{T}}$ is a CW complex and identify

$$H^n_{\mathbb{T}}(X, F; \mathbb{Q}) = H^n(X_{\mathbb{T}}, F_{\mathbb{T}}; \mathbb{Q}) = H^n(p^{-1}(B^k_{\mathbb{T}}), F_{\mathbb{T}} \cap p^{-1}(B^k_{\mathbb{T}}); \mathbb{Q}) \quad \text{for } k > n.$$

Since $p^{-1}(B^k_{\mathbb{T}})$ is compact, so is $F_{\mathbb{T}} \cap p^{-1}(B^k_{\mathbb{T}})$. Therefore by the continuity of cohomology there exists an invariant neighbourhood V of F such that

$$\phi \in \mathrm{Im}(H^*_{\mathbb{T}}(X, V; \mathbb{Q}) \longrightarrow H^*_{\mathbb{T}}(X, \mathrm{int} V; \mathbb{Q}) \longrightarrow H^*_{\mathbb{T}}(X, F; \mathbb{Q}))\,.$$

On the other hand, there exists an invariant compact space $Y \subseteq X - F$ such that $\mathrm{int} V \cup \mathrm{int} Y = X$. By the first part of the proof there exists a positive integer q so that u^q maps to $0 \in H^*_{\mathbb{T}}(Y; \mathbb{Q})$. By the exact sequence of the pair (X, Y) and the relative cup product $u^q.\phi$ lies in the image of

$$0 = H^*_{\mathbb{T}}(X, \mathrm{int} V \cup \mathrm{int} Y; \mathbb{Q}) \longrightarrow H^*_{\mathbb{T}}(X, F, \mathbb{Q})$$

and so $u^q.\phi = 0$. $\boxed{1.4.3}$

1.5. Infinite dimensions

There are interesting infinite dimensional spaces with natural circle actions e.g. the free loop space LM of a manifold M with the circle action being the rotations. In this case the fixed point set is exactly the underlying manifold M. However, in [Goodwillie 85] it is shown that $u^{-1}H^*_{\mathbb{T}}(LM; \mathbb{Q})$ depends only on the fundamental group of M. Therefore the fixed point theorem does not hold for localised equivariant cohomology of infinite dimensional $\mathbb{T}$-spaces. The problem is that $u^{-1}H^*_{\mathbb{T}}$ does not satisfy Milnor's additivity axiom which is of particular importance for the cohomology of infinite dimensional spaces [Milnor 62], [Whitehead 78]. A cohomology theory h^* satisfies Milnor's additivity axiom if

$$h^*(\coprod_i X_i) \cong \prod_i h^*(X_i)$$

for any family of objects X_i such that $\coprod_i X_i$ is an object in the category on which h^* is defined. Borel equivariant cohomology satisfies this axiom, because singular cohomology does. The localisation operator, however, in general does not commute with infinite products.

Therefore localised equivariant cohomology does not satisfy Milnor's axiom. A counterexample is the disjoint union of even dimensional spheres with the trivial action. The equivariant cohomology of S^n with the trivial action is

$$H_{\mathbf{T}}^*(S^n) \cong (\mathbf{Z} \oplus \mathbf{Z}\sigma_n) \otimes \mathbf{Z}[u]$$

where $\sigma_n \in H^n(S^n)$ is the dual of the fundamental class of S^n. The image of the natural homomorphism

$$u^{-1}\Big(\prod_{n \in \mathbf{N}} H_{\mathbf{T}}^*(S^{2n})\Big) \longrightarrow \prod_{n \in \mathbf{N}} u^{-1} H_{\mathbf{T}}^*(S^{2n})$$

does not contain the element $\sum u^{-n}\sigma_{2n}$.

Note that the cohomology

$$X \mapsto \begin{cases} \prod_k H^{2k}(X^{\mathbf{T}}; \mathbf{Q}) & \text{in every even degree} \\ \prod_k H^{2k+1}(X^{\mathbf{T}}; \mathbf{Q}) & \text{in every odd degree} \end{cases}$$

agrees with $u^{-1}H_{\mathbf{T}}^*(-; \mathbf{Q})$ on compact finite dimensional spaces and satisfies Milnor's additivity axiom. Due to the strong uniqueness properties of such cohomologies it is reasonable to expect that the equivariant cohomology theory we want to construct does satisfy the fixed point theorem, provided that the coefficients are in a field of characteristic zero.

1.6. Jones-Petrack theory

In [Jones, Petrack 88] the de Rham version of localised equivariant cohomology in infinite dimensions is modified to satisfy Milnor's axiom and the fixed point theorem in the following way. Because localisation and homology commute

$$u^{-1}H_{\mathbf{T}}^*(X; \mathbf{C}) = \mathrm{H}(\Omega_{inv}^*(X)[u, u^{-1}]; d_{\mathbf{T}}) .$$

Localised equivariant cohomology does not satisfy Milnor's additivity axiom, because the cochain complex involves polynomials in u, u^{-1}. Jones and Petrack replaced the polynomials in u, u^{-1} by infinite power series and thus obtained an equivariant cohomology theory satisfying Milnor's axiom and coinciding with localised equivariant cohomology on finite dimensional smooth $\mathbf{T}$-manifolds.

Let X be a smooth $\mathbf{T}$-manifold which admits smooth partitions of unity and a smoothly varying positive definite inner product (that is a pre-Hilbert space structure) on each tangent space. Let $\Omega_{inv}^*(X)[[u^{-1}, u]$ denote the set of formal power series in u, u^{-1} with coefficients in $\Omega_{inv}^*(X)$ in the following sense: A typical homogeneous element ω, of total degree p, in $\Omega_{inv}^*(X)[[u^{-1}, u]$ has the form

$$\omega = \sum_{-\infty < i}^{p/2} \alpha_i u^i , \quad \alpha_i \in \Omega_{inv}^{p-2i}(X) .$$

If we denote by $(\Omega^*_{inv}(X)[[u^{-1},u])^p$ the set of all homogeneous elements of degree p, then

$$\Omega^*_{inv}(X)[[u^{-1},u] = \bigoplus_{p\in\mathbb{Z}}(\Omega^*_{inv}(X)[[u^{-1},u])^p .$$

The homogeneous elements can be multiplied, so $\Omega^*_{inv}(X)[[u^{-1},u]$ becomes a graded ring and $d_{\mathbf{T}} = d + \iota_V u$ is a derivation with respect to this product. Jones and Petrack defined the completed periodic equivariant cohomology $h^*_{\mathbf{T}}(X)$ to be

$$h^*_{\mathbf{T}}(X) = \mathrm{H}(\Omega^*_{inv}(X)[[u^{-1},u]; d_{\mathbf{T}}) .$$

Because of

$$\Omega^*_{inv}(\coprod_i X_i)[[u^{-1},u] = \prod_i \Omega^*_{inv}(X_i)[[u^{-1},u]$$

$h^*_{\mathbf{T}}$ satisfies Milnor's axiom. It agrees with localised equivariant cohomology on all finite dimensional smooth $\mathbf{T}$-manifolds because Ω^*_{inv} is bounded above in this case. There is also a fixed point theorem for $h^*_{\mathbf{T}}$. Let X be a $\mathbf{T}$-manifold with a smooth partition of unity and a smoothly varying positive definite inner product and such that $F = X^{\mathbf{T}}$ has an invariant tubular neighbourhood.

Lemma 1.6.1. *Let X be a smooth $\mathbf{T}$-manifold without fixed points. then*

$$h^*_{\mathbf{T}}(X) = 0 .$$

Proof of 1.6.1: Because $\Omega^*_{inv}(X)[[u^{-1},u]$ has a product and the differential $d_{\mathbf{T}}$ satisfies the Leibniz rule with respect to this product, it suffices to find an element $\omega \in \Omega^*_{inv}(X)[[u^{-1},u]$ such that $d_{\mathbf{T}}\omega = 1$. Let V be the vector field on X generating the circle action. Because there are no fixed points this vector field is nowhere zero. Therefore there exists an invariant differential form α dual to V. Then

$$d_{\mathbf{T}}\alpha = d\alpha + |V|^2 u = |V|^2 u \left(1 + \frac{d\alpha}{|V|^2 u}\right).$$

Therefore $d_{\mathbf{T}}\alpha$ is invertible in $\Omega^*_{inv}(X)[[u^{-1},u]$, its inverse is given by

$$\gamma = (d_{\mathbf{T}}\alpha)^{-1} = |V|^{-2} u^{-1} \sum_{i=0}^{\infty} \left(\frac{-d\alpha}{|V|^2}\right)^i u^{-i} .$$

Since $\gamma(d_{\mathbf{T}}\alpha) = 1$ it follows that $(d_{\mathbf{T}}\gamma)(d_{\mathbf{T}}\alpha) = 0$ and therefore, since $d_{\mathbf{T}}\alpha$ is invertible, $d_{\mathbf{T}}\gamma = 0$. So $d_{\mathbf{T}}(\alpha\gamma) = (d_{\mathbf{T}}\alpha)\gamma = 1$ and we define $\omega = \alpha\gamma$. $\boxed{1.6.1}$

Theorem 1.6.2. *The inclusion $i : F \hookrightarrow X$ induces an isomorphism*

$$i^* : h^*_{\mathbf{T}}(X) \longrightarrow h^*_{\mathbf{T}}(F) .$$

Proof of 1.6.2: We use the Mayer-Vietoris sequence associated to $X - F$ and the invariant tubular neighbourhood N of F. Both $X - F$ and $(X - F) \cap N$ are fixed point free and so, by the lemma above, we obtain the isomorphism. $\boxed{1.6.2}$

Chapter 2

Two counterexamples

Here we discuss two obvious candidates for a singular version of the completed periodic $\mathbf{T}$-equivariant cohomology $h_{\mathbf{T}}^{*}$ of [Jones, Petrack 88] and show that they do not have the required properties. We are looking for an equivariant cohomology which agrees with localised equivariant cohomology on compact spaces but satisfies Milnor's additivity axiom. Milnor's axiom forces us to replace the complex $S^*(X)[u^{-1}, u]$ by $S^*(X)[[u^{-1}, u]$. For finite dimensional spaces this looks innocuous since $S^*(X)$ has finite cohomological dimension in this case, but in fact it does change cohomology as we shall show. The cochain complex $S^*(X)[[u^{-1}, u]$ is in fact acyclic for any $\mathbf{T}$-space X. If we replace $S^*(X)$ by normalised cochains $C^*(X)$, we obtain a theory which agrees with localised equivariant cohomology on spaces with trivial action. However, we show that it is trivial on all fixed-point free $\mathbf{T}$-spaces and therefore does not agree with $u^{-1}H_{\mathbf{T}}^{*}(-;\mathbf{Z})$ even on homogeneous $\mathbf{T}$-spaces.

2.1. Preliminaries on singular cochains

Here we detail some basic definitions to check that all the necessary operations are well defined.

There exists exactly one affine map from the standard $(n+1)$-simplex Δ^{n+1} to the standard n-simplex Δ^n which maps the first $k+1$ vertices of Δ^{n+1} into the first $k+1$ vertices of Δ^n $(0 \leq k \leq n)$ and vertices $k+1, \ldots, n+1$ into the vertices $k, \ldots, n$ of Δ^n. This map is called the k^{th} degeneracy of Δ^{n+1} onto Δ^n. This map gives rise to the k^{th} degeneracy operator $s_k : S_n(X) \to S_{n+1}(X)$ which maps the singular n-simplexes in a space X to the singular $(n+1)$-simplexes in X. Singular simplexes which are in the image of any degeneracy operator are called degenerate.

There is a natural homomorphism

$$\mathrm{sh} : S_*(X) \otimes S_*(Y) \longrightarrow S_*(X \times Y)$$

called the shuffle product, defined for all spaces X, Y. It is defined as follows: A (p,q)-shuffle in $(0,\ldots,p+q-1)$ is a permutation $\epsilon = (i_1,\ldots,i_p,j_1,\ldots,j_q)$ of $(0,\ldots,p+q-1)$ such that $i_1 < i_2 < \ldots < i_p$, $j_1 < j_2 < \ldots < j_q$. Then

$$\mathrm{sh}(\sigma \otimes \tau) = \sum_{\epsilon} \mathrm{sign}(\epsilon)(s_{j_q}\ldots s_{j_1}\sigma, s_{i_p}\ldots s_{i_1}\tau)$$

where $\sigma \in S_p(X)$, $\tau \in S_q(Y)$, the sum is over all (p,q)-shuffles ϵ of the ordered set $(0,\ldots,p+q-1)$.

Geometrically we can think of $\mathrm{sh}(\sigma \otimes \tau)$ as the maps σ and τ applied on a canonical triangulation of $\Delta^p \times \Delta^q$.

The shuffle product is associative. The shuffle product of singular simplexes, at least one of which is degenerate, is a sum of degenerate simplexes.

In the unit circle $\mathbf{T} \subset \mathbf{C}$ the shuffle product together with the multiplication map give us a product

$$S_*(\mathbf{T}) \otimes S_*(\mathbf{T}) \longrightarrow S_*(\mathbf{T} \times \mathbf{T}) \longrightarrow S_*(\mathbf{T})\,.$$

This product makes $S_*(\mathbf{T})$ into a commutative differential graded algebra. That is, $S_*(\mathbf{T})$ is a differential graded module with a product which is associative and graded commutative, addition and multiplication are related by distributivity and the differential map satisfies the Leibniz rule.

Let X be a space with a $\mathbf{T}$-action $\Phi : \mathbf{T} \times X \longrightarrow X$. The singular cochain complex $S^*(X;R)$ of X with coefficients in a ring R (which will be fixed and so usually dropped from the notation) becomes a right $S_*(\mathbf{T})$-module with the product given by

$$< \phi.\sigma, \tau > = < \Phi^*\phi/\sigma, \tau > = < \phi, \Phi\mathrm{sh}(\sigma \otimes \tau) >$$

for $\sigma \in S_p(\mathbf{T}), \phi \in S^q(X), \tau \in S_{q-p}(X)$, and is trivial for $q < p$. Our conventions for the slant product are

$$< \alpha/\sigma, \tau > = < \alpha, \mathrm{sh}(\sigma \otimes \tau) >$$

for $\alpha \in S^*(X \times Y), \sigma \in S_*(X), \tau \in S_*(Y)$. The shuffle product commutes with the boundary operator ∂ if we define $\partial(x \otimes y) = \partial x \otimes y + (-1)^{|x|}x \otimes \partial y$. The coboundary operator acts on the slant product by

$$\begin{aligned}
< \delta(\alpha/\sigma), \tau > &= < \alpha/\sigma, \partial\tau > = < \alpha, \mathrm{sh}(\sigma \otimes \partial\tau) > \\
&= (-1)^{|\sigma|} < \alpha, \partial(\mathrm{sh}(\sigma \otimes \tau)) - \mathrm{sh}(\partial\sigma \otimes \tau) > \\
&= (-1)^{|\sigma|} < (\delta\alpha)/\sigma - \alpha/(\partial\sigma), \tau >\,.
\end{aligned}$$

The shuffle homomorphism is functorial, i.e. it commutes with maps of spaces. Therefore a $\mathbf{T}$-equivariant map $f : X \to Y$ induces a cochain homomorphism

$f^* : S^*(Y) \to S^*(X)$ which is also a homomorphism of $S_*(\mathbf{T})$-modules. Because the shuffle product is degenerate if one of the factors is degenerate, the cochain complex

$$C^*(X) = \mathrm{Hom}_R(S_*(X)/D_*(X); R),$$

where $D_*(X)$ is the subcomplex of the degenerate simplexes, is a right $S_*(\mathbf{T})$-module. We call the elements of $C^*(X)$ normalised singular cochains on X.

There is a natural product

$$\cup : S^*(X) \otimes S^*(X) \longrightarrow S^*(X)$$

called the cup product, defined for all spaces X. It is defined as follows. For a singular n-simplex σ in X and $0 \le i \le n$, let $\mathrm{Fr}^i\sigma \in S_i(X)$ be the map σ restricted to the front i-face of Δ^n and $\mathrm{Bk}^i\sigma \in S_i(X)$ be the map σ restricted to the back i-face of Δ^n. The cup product is defined to be bilinear and for $\alpha \in S^p(X)$, $\beta \in S^q(X)$ the value of their product $\alpha \cup \beta$ on a singular $(p+q)$-simplex σ is defined to be

$$< \alpha \cup \beta, \sigma > = < \alpha, \mathrm{Fr}^p\sigma > . < \beta, \mathrm{Bk}^q\sigma > .$$

The cup product is extended linearly to obtain an element in $S^{p+q}(X)$. Notice that for every degenerate singular $(p+q)$-simplex σ in X, at least one of $\mathrm{Fr}^p\sigma$ and $\mathrm{Bk}^q\sigma$ is degenerate and thus we obtain also a product

$$\cup : C^p(X) \times C^q(X) \longrightarrow C^{p+q}(X).$$

2.2. The I operator

Definition 2.2.1. Let X be a topological space with a left $\mathbf{T}$ action $\Phi : \mathbf{T} \times X \to X$. Define the operator I: $S^n(X) \to S^{n-1}(X)$ by

$$\mathrm{I}(x) = x.v = \Phi^* x / v$$

where v is the singular 1-simplex in $\mathbf{T}$ given by $v : [0,1] \to \mathbf{T}, v(t) = \exp(2\pi i t)$. This operator induces an operator I: $C^n(X) \to C^{n-1}(X)$.

The operator I is well-defined and functorial with respect to $\mathbf{T}$-equivariant maps.

Proposition 2.2.2. $\mathrm{I}^2 = 0$.

Proof of 2.2.2: Since both $S^*(X)$ and $C^*(X)$ are $S_*(\mathbf{T})$-modules, it suffices to show that $v^2 = 0 \in S_2(\mathbf{T})$. The multiplication in $S_*(\mathbf{T})$ is given by

$$S_*(\mathbf{T}) \otimes S_*(\mathbf{T}) \xrightarrow{\mathrm{sh}} S_*(\mathbf{T} \times \mathbf{T}) \xrightarrow{\mu} S_*(\mathbf{T})$$

where $\mu : S_n(\mathbf{T}\times\mathbf{T}) \to S_n(\mathbf{T})$ is induced by the multiplication in $\mathbf{T}$. So

$$\begin{aligned} v.v &= \mu\,\mathrm{sh}(v\otimes v) = -\mu(s_0v, s_1v) + \mu(s_1v, s_0v) \\ &= -\exp(2\pi i(x+y)) + \exp(2\pi i(y+x)) = 0\ . \end{aligned}$$

$\boxed{2.2.2}$

Proposition 2.2.3. $\delta\mathrm{I} + \mathrm{I}\delta = 0$.

Proof of 2.2.3:

$$\partial(v\otimes\sigma) = \partial v\otimes\sigma - v\otimes\partial\sigma = -v\otimes\partial\sigma$$

as $\partial v = 0$ and so

$$\begin{aligned} <\delta(\mathrm{I}\alpha),\sigma> &= <\delta(\Phi^*\alpha/v),\sigma> = <\Phi^*\alpha/v,\partial\sigma> = <\Phi^*\alpha,\mathrm{sh}(v\otimes\partial\sigma)> \\ &= <\Phi^*\alpha,-\partial\mathrm{sh}(v\otimes\sigma)> = -<\delta\Phi^*\alpha,\mathrm{sh}(v\otimes\sigma)> \\ &= -<\Phi^*\delta\alpha,\mathrm{sh}(v\otimes\sigma)> = <-\mathrm{I}(\delta\alpha),\sigma> \end{aligned}$$

whence the result. $\boxed{2.2.3}$

Lemma 2.2.4. *Let X be a $\mathbf{T}$-space and $\alpha\in S^p(X)$, $\beta\in S^q(X)$. Then*

$$\mathrm{I}(\alpha\cup\beta) = \mathrm{I}(\alpha)\cup\beta + (-1)^p\alpha\cup\mathrm{I}(\beta)\ .$$

Proof of 2.2.4: Let $\sigma\in S_{p+q-1}(X)$. Define $(v,\sigma)_i$ to be the singular simplex $(s_{p+q-1}\dots\hat{s}_i\dots s_0v, s_i\sigma)$ in $\mathrm{sh}(v,\sigma)$. Then

$$<\mathrm{I}(\alpha\cup\beta),\sigma> = <\alpha\cup\beta,\Phi\mathrm{sh}(v,\sigma)> =$$

$$\sum_{i=0}^{p+q-1}(-1)^i<\alpha,\Phi\mathrm{Fr}^p(v,\sigma)_i><\beta,\Phi\mathrm{Bk}^q(v,\sigma)_i>\ .$$

Notice that all the simplexes in the shuffle product for which $i\le p-1$ have $\mathrm{Bk}^q\sigma$ as their back q-face; similarly, all the simplexes with $i\ge p$ have $\mathrm{Fr}^p\sigma$ as their front p-face. Therefore

$<\mathrm{I}(\alpha\cup\beta),\sigma> =$

$$\sum_{i=0}^{p-1}(-1)^i<\alpha,\Phi(v,\mathrm{Fr}^{p-1}\sigma)_i><\beta,\mathrm{Bk}^q\sigma>$$

$$+\sum_{i=p}^{p+q-1}(-1)^i<\alpha,\mathrm{Fr}^p\sigma><\beta,\Phi(v,\mathrm{Bk}^{q-1}\sigma)_i>$$

$$=< \alpha, \Phi\mathrm{sh}(v, \mathrm{Fr}^{p-1}\sigma) >< \beta, \mathrm{Bk}^q\sigma > +(-1)^p < \alpha, \mathrm{Fr}^p\sigma >< \beta, \Phi\mathrm{sh}(v, \mathrm{Bk}^{q-1}\sigma) >$$

$$=< \mathrm{I}\alpha, \mathrm{Fr}^{p-1}\sigma >< \beta, \mathrm{Bk}^q\sigma > +(-1)^p < \alpha, \mathrm{Fr}^p\sigma >< \mathrm{I}\beta, \mathrm{Bk}^{q-1}\sigma >$$

$$=< \mathrm{I}(\alpha) \cup \beta > +(-1)^p < \alpha \cup \mathrm{I}(\beta), \sigma > .$$

$\boxed{2.2.4}$

2.3. Counterexamples

Definition 2.3.1. Define a cochain complex $(S^*(X)[[u^{-1}, u]; \delta_T)$ as follows. Let u be an indeterminate of degree 2 and define $(S^*(X)[[u^{-1}, u])^p$ to be the set of the formal power series in u^{-1}, u with coefficients in $S^*(X)$ which are of the form

$$\omega = \sum_{-\infty}^{p} a_i u^i , \quad a_i \in S^{p-2i}(X).$$

Now define

$$S^*(X)[[u^{-1}, u] = \bigoplus_{n \in \mathbb{Z}} (S^*(X)[[u^{-1}, u])^n$$

and $\delta_T = \delta + \mathrm{I}u$. Then $S^*(X)[[u^{-1}, u]$ is a cochain complex with the differential map δ_T.

Contrary to the de Rham case of [Jones, Petrack 88] the homology of this cochain complex differs from the homology of $S^*(X)[u^{-1}, u]$ even for $X = \mathrm{pt}$ and is therefore not a suitable extension of localised equivariant cohomology.

Example 2.3.2.

$$\mathrm{H}(S^*(\mathrm{pt}; \mathbf{Z})[[u^{-1}, u]) = 0 .$$

Proof of 2.3.2: Let us denote the singular n-cochain which maps the constant n-simplex into $1 \in \mathbf{Z}$ by f_n. Then the operators δ and I act on cochains by

$$\delta(f_n) = \begin{cases} f_{n+1} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$$\mathrm{I}(f_n) = \begin{cases} f_{n-1} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Given a 0-cochain

$$\sum_{0}^{\infty} n_{2i} f_{2i} u^{-i}$$

where $n_i \in \mathbb{Z}$, define $m_{2i+1} = \sum_0^i (-1)^j n_{2j}$. Then we obtain

$$\delta_T(\sum_0^\infty m_{2i+1} f_{2i+1} u^{-i+1}) = \sum_0^\infty n_{2i} f_{2i} u^{-i} .$$

Because of the periodicity this implies that the cohomology is trivial in the even degrees. It is easy to see that 0 is the only cocycle of odd degree. $\boxed{2.3.2}$

In the same way we can prove $\mathrm{H}(S^*(\mathrm{pt};\mathbb{C})[[u^{-1},u]) = 0$. Therefore

$$\mathrm{H}(S^*(\mathrm{pt};\mathbb{C})[[u^{-1},u]) \not\cong \mathrm{H}(\Omega^*(\mathrm{pt};\mathbb{C})[[u^{-1},u]).$$

In case $X = \mathrm{pt}$ we can obtain the desired result if we replace singular cochains by normalised singular cochains. However, we show later that also this modification does not give a suitable extension of localised equivariant cohomology.

Definition 2.3.3. Define a cochain complex $(C^*(X)[[u^{-1},u];\delta_T)$ to be the subcomplex of $S^*(X)[[u^{-1},u]$ of those formal power series in u^{-1}, u which have all their coefficients in $C^*(X)$.

Products can be extended to $C^*(X)[[u^{-1},u]$ in a straightforward manner as every coefficient in the product series is a finite sum of the coefficients in the factors.

Corollary 2.3.4. *The differential δ_T in the graded ring $C^*(X)[[u^{-1},u]$ satisfies the Leibniz rule.*

Proof of 2.3.4: This is an immediate consequence of Lemma 2.2.4 and the fact that the operator δ satisfies the Leibniz rule. $\boxed{2.3.4}$

Proposition 2.3.5.

$$H^*(C^*(X;R)[[u^{-1},u];\delta_T) = H^{*+2}(C^*(X;R)[[u^{-1},u];\delta_T)$$

Proof of 2.3.5: There is an isomorphism of cochain complexes

$$(C^*(X)[[u^{-1},u])^n \longrightarrow (C^*(X)[[u^{-1},u])^{n+2}$$

which maps $\sum \alpha_i u^i \mapsto \sum \alpha_i u^{i+1}$. $\boxed{2.3.5}$

Because of $C^n(\mathrm{pt}) = 0$ for $n > 0$ we obtain

Example 2.3.6.

$$\mathrm{H}(C^*(\mathrm{pt};R)[[u^{-1},u];\delta_T) = \begin{cases} R[u,u^{-1}] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

In other words $\mathrm{H}(C^*(\mathrm{pt}; R)[[u^{-1}, u]; \delta_{\mathbf{T}}) \cong u^{-1}H^*_{\mathbf{T}}(\mathrm{pt}; R)$.

Next we show that even this modified complex of power series in u^{-1}, u does not agree with localised equivariant cohomology on homogeneous $\mathbf{T}$-spaces. In Chapter 1 we have seen that $u^{-1}H^*_{\mathbf{T}}(X; \mathbf{Z}) \neq 0$, if X is a homogeneous $\mathbf{T}$ space such that $\mathbf{T}$ does not act freely on it.

Proposition 2.3.7. *If $X^{\mathbf{T}} = \emptyset$, then*

$$H(C^*(X; R)[[u^{-1}, u]; \delta_{\mathbf{T}}) = 0 .$$

Proof of 2.3.7: Define $\alpha \in C^1(X)$ by

$$< \alpha, \sigma > = \begin{cases} 1 & \text{if } \sigma(t) = \Phi(t, \sigma(0)) \\ 0 & \text{otherwise} \end{cases}$$

for $\sigma \in S_1(X)$. By construction we have $\mathrm{I}(\alpha) = 1 \in C^0(X)$. By the Leibniz rule we have

$$\mathrm{I}(\alpha \cup (\delta\alpha)^n) = (\delta\alpha)^n$$
$$\delta(\alpha \cup (\delta\alpha)^n) = (\delta\alpha)^{n+1}$$

where $(\delta\alpha)^n$ denotes the n-fold cup product of $\delta\alpha$. The (-1)-cochain

$$\gamma = \sum_{n\geq 0}(-1)^n \alpha \cup (\delta\alpha)^n u^{-n-1} \quad \in C^*(X)[[u^{-1}, u]$$

is such that $\delta_{\mathbf{T}}\gamma = 1 \in C^*(X)[[u^{-1}, u]$. For $\omega \in C^*(X)[[u^{-1}, u]$ such that $\delta_{\mathbf{T}}\omega = 0$ we have

$$\delta_{\mathbf{T}}(\gamma\omega) = \omega.$$

$\boxed{2.3.7}$

Chapter 3

Local approach

As we have seen in the previous chapter the naive approach to extend localised equivariant cohomology to infinite dimensional spaces by using formal power series fails. We need some sort of Mayer-Vietoris principle to re-assemble the local cohomological information to obtain a global cohomology for which the fixed point theorem is true. The method of doing this is to reformulate localised equivariant cohomology on compact $\mathbb{T}$-pairs as the cohomology of appropriate differential graded sheaves and extend this theory to infinite dimensional spaces in a suitable way to obtain some sort of Mayer-Vietoris principle for DG sheaves. This can be done in a standard manner if the DG sheaves are bounded below, e.g. [Griffiths, Harris 78], but the DG sheaves which we obtain are not bounded below. Therefore we have to modify the standard DG sheaf cohomology and this is what we do in the next chapter. It turns out that the modified DG sheaf cohomology gives the correct fixed point theorem as well as Milnor's additivity property.

In this chapter we define Čech cohomology of DG presheaves in Section 3.2. In Section 3.3 we show that localised equivariant cohomology of "nice" $\mathbb{T}$-pairs is isomorphic to the Čech cohomology of certain DG presheaves. In Section 3.4 we assume there exists a completed cohomology theory of DG sheaves with certain properties which we list and we define Jones-Petrack cohomology using this completed cohomology of DG sheaves. Then we show how all the desired properties of Jones-Petrack cohomology are a formal consequence of the properties of the complete cohomology of DG sheaves which is constructed in Chapter 4.

We think of sheaves as presheaves with special properties. All spaces we consider are assumed to be metrisable.

3.1. Preliminaries on presheaves and sheaves

Definition 3.1.1. Let R be a ring. A *presheaf* $\mathcal{L}$ of R-modules on a space X is a contravariant functor from the category of open subsets of X and inclusions to the category of R-modules.

We only consider presheaves $\mathcal{L}$ such that $\mathcal{L}(\emptyset) = 0$.

Definition 3.1.2. Let $\mathcal{L}$ and $\mathcal{M}$ be two presheaves on a space X. A *presheaf homomorphism* $k : \mathcal{L} \to \mathcal{M}$ is a natural transformation of functors.

Definition 3.1.3. Let $\mathcal{L}$ be a presheaf on a space X and $\mathcal{M}$ a presheaf on a space Y. Let $f : X \to Y$ be a map. Then a *presheaf homomorphism k over f* is a collection of homomorphisms $k_U : \mathcal{M}(U) \to \mathcal{L}(f^{-1}U)$, one for each open subset U in Y, such that for all open sets $V \subset U$ the diagrams

$$\begin{array}{ccc} \mathcal{M}(U) & \xrightarrow{k_U} & \mathcal{L}(f^{-1}U) \\ \downarrow & & \downarrow \\ \mathcal{M}(V) & \xrightarrow{k_V} & \mathcal{L}(f^{-1}V) \end{array}$$

commute. Here the vertical arrows are restriction homomorphisms.

Definition 3.1.4. Let X be a topological space and R a ring. A *differential graded presheaf* $(\mathcal{L}^\bullet, d)$ of R-modules on X is a sequence $\mathcal{L}^n, n \in \mathbb{Z}$ of presheaves of R-modules together with a sequence of presheaf homomorphisms $d : \mathcal{L}^n \to \mathcal{L}^{n+1}$ such that $dd = 0$. A *DG presheaf homomorphism* $k^\bullet : \mathcal{L}^\bullet \to \mathcal{M}^\bullet$ is a sequence of presheaf homomorphisms $k^n : \mathcal{L}^n \to \mathcal{M}^n$ which commute with the differentials in $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$, respectively. If $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ are DG presheaves on X and Y, respectively, a *DG presheaf homomorphism $k^\bullet$ over f* is a collection $k^n : \mathcal{M}^n \to \mathcal{L}^n$ of homomorphisms of presheaves over f which commute with the differential morphisms in the respective DG presheaves.

Definition 3.1.5. Let $\mathcal{L}$ be a presheaf of R-modules on X and x a point in X. Then the R-module

$$\mathcal{L}_x = \varinjlim_{U \ni x} \mathcal{L}(U)$$

is called the *stalk* of $\mathcal{L}$ at x.

The stalk of a DG presheaf $\mathcal{L}^\bullet$ at x is the sequence $\mathcal{L}^n_x$ together with the induced homomorphisms $d : \mathcal{L}^n_x \to \mathcal{L}^{n+1}_x$.

Let $f : X \to Y$ be a map, $\mathcal{A}$ a presheaf on X, $\mathcal{B}$ a presheaf on Y and $k : \mathcal{B} \to \mathcal{A}$ a presheaf homomorphism over f. Then k induces a family of homomorphisms

$$k_x : \mathcal{B}_{f(x)} \longrightarrow \mathcal{A}_x$$

for every $x \in X$ obtained by

$$\lim_{V \ni f(x)} \mathcal{B}(V) \xrightarrow{k_V} \lim_{f^{-1}V \ni x} \mathcal{A}(f^{-1}V) \longrightarrow \lim_{U \ni x} \mathcal{A}(U).$$

Definition 3.1.6. Let X be a topological space and R a ring. A presheaf $\mathcal{L}$ of R-modules on X is a *sheaf* if it satisfies the following two properties.

- Zero is the only locally trivial element: Let $\{V_i; i \in J\}$ be a family of open sets with $\bigcup_i V_i = U$ and let $\gamma \in \mathcal{L}(U)$ be such that $\gamma|_{V_i} = 0$ for every $i \in J$. Then $\gamma = 0 \in \mathcal{L}(U)$.
- Collation property: Let $\{V_i; i \in J\}$ be a family of open sets with $\bigcup_i V_i = U$ and let $\gamma_i \in \mathcal{L}(V_i)$ be such that $\gamma_i|_{V_i \cap V_j} = \gamma_j|_{V_i \cap V_j}$ for every pair $i, j \in J$. Then there exists an element $\gamma \in \mathcal{L}(U)$ such that $\gamma|_{V_i} = \gamma_i$.

Definition 3.1.7. A *differential graded sheaf* $(\mathcal{L}^\bullet, d)$ of R-modules on X is a DG presheaf of R-modules such that every presheaf $\mathcal{L}^n$, $n \in \mathbb{Z}$, is a sheaf.

Every presheaf canonically induces a sheaf which is usually called the associated sheaf of the presheaf. This is done in the following way. Let $\mathcal{L}$ be a presheaf on a space X. For an element $t \in \mathcal{L}(U)$ let $[t]_x \in \mathcal{L}_x$, for $x \in U$, be the element in the stalk of $\mathcal{L}$ over x, determined by the direct limit of the restrictions of t. For every open subset U of X define an R-module $\mathcal{L}^+(U)$ to be the set of functions

$$s : U \mapsto \coprod_{x \in U} \mathcal{L}_x$$

such that

- $s(x) \in \mathcal{L}_x$ for every $x \in U$;
- for every $x \in U$, there exists an open neighbourhood V of x, $V \subseteq U$, and an element $t \in \mathcal{L}(V)$ such that $s(y) = [t]_y \in \mathcal{L}_y$ for every $y \in V$.

The R-modules $\mathcal{L}^+(U)$, for all open subsets U in X, and the natural restrictions define a sheaf $\mathcal{L}^+$ on X, called the sheaf associated to the presheaf $\mathcal{L}$.

Every presheaf homomorphism over a map induces a sheaf homomorphism of the associated sheaves over the same map. The same is true for DG presheaves, DG sheaves and their homomorphisms.

We do not consider the categories of sheaves as subcategories of presheaves, however. The following definitions for sheaves do not agree with the corresponding ones for presheaves.

Definition 3.1.8. Let $\mathcal{L}$ be a subsheaf of a sheaf $\mathcal{M}$. The *quotient sheaf* $\mathcal{Q}$ is the sheaf associated to the presheaf

$$U \mapsto \mathcal{M}(U)/\mathcal{L}(U).$$

Definition 3.1.9. Let us call a sequence of presheaf homomorphisms on X *exact* if it gives an exact sequence of module homomorphisms for each open set of X. We call a sequence of sheaf homomorphisms *exact* if the induced sequences of stalks are all exact sequences of modules.

There is another characterisation of sheaf homomorphisms over maps.

Definition 3.1.10. Let $f : X \to Y$ be a map and $\mathcal{L}$ be a sheaf on Y. Then $f^{!}\mathcal{S}$, the *pull-back* of $\mathcal{L}$ by f, is the sheaf associated to the presheaf on X defined by

$$U \mapsto \lim_{V \supseteq f(U)} \mathcal{L}(V)$$

for U open subsets of X, where the limit is over all open sets V in Y containing $f(U)$. The pull-back of a DG sheaf is defined similarly and is again a DG sheaf.

Let $f : X \to Y$ and let $\mathcal{L}$ and $\mathcal{M}$ be sheaves on X and Y, respectively. Then there is a bijective correspondence between the sheaf homomorphisms $\mathcal{M} \to \mathcal{L}$ over f and the homomorphisms $f^{!}\mathcal{M} \to \mathcal{L}$ of sheaves on X. The same is true for DG sheaves and DG sheaf homomorphisms.

We shall need also the following notions associated to a DG sheaf.

Definition 3.1.11. Let $\mathcal{L}^{\bullet}$ be a DG sheaf and let $\mathcal{K}^n = \mathrm{Ker}(d : \mathcal{L}^n \to \mathcal{L}^{n+1})$ and $\mathcal{B}^n = \mathrm{Im}(d : \mathcal{L}^{n-1} \to \mathcal{L}^n)$. The sheaf $\mathcal{H}^n$ associated to the quotient presheaf $\mathcal{K}^n/\mathcal{B}^n$ is called the *cohomology sheaf* of degree n of $\mathcal{L}^{\bullet}$. We call $\mathcal{H}^{\bullet}$, with the trivial differential, the *cohomology DG sheaf* of $\mathcal{L}^{\bullet}$.

The stalk of the cohomology DG sheaf $\mathcal{H}^{\bullet}$ of a DG sheaf $\mathcal{L}^{\bullet}$ at any point $x \in X$ is

$$\mathcal{H}^{\bullet}_x = \mathrm{H}(\mathcal{L}^{\bullet}_x).$$

Definition 3.1.12. Let $\mathcal{L}^{\bullet}$ and $\mathcal{M}^{\bullet}$ be DG sheaves on X and $\phi : \mathcal{L}^{\bullet} \to \mathcal{M}^{\bullet}$ a DG sheaf homomorphism so that ϕ induces an isomorphism of the respective cohomology DG sheaves. Then we call ϕ a *quasi-isomorphism* of DG sheaves.

Definition 3.1.13. The *cohomological dimension*, $\mathrm{cd}X$, of a space X over a ring R is defined in the following way. We say $\mathrm{cd}X \leq n$ if $H^m(X;\mathcal{L}) = 0$ for any sheaf $\mathcal{L}$ of R-modules on X and any positive integer $m > n$. We say that a pair (X, A) has finite cohomological dimension if the space X has finite cohomological dimension.

Every subspace of a metrisable space with finite cohomological dimension has finite cohomological dimension [Godement 58, p. 196]. If a $\mathbf{T}$-space has finite cohomological dimension, so has its orbit space [Quillen 71, p. 572].

3.2. Čech cohomology of DG presheaves

In this section we define Čech cohomology of DG presheaves. We prove several properties of Čech cohomology in Appendix A.

Definition 3.2.1. A *double complex* C is a family

$$\{C^{p,q}; p, q \in \mathbf{Z}\}$$

of modules with two families

$$\delta : C^{p,q} \to C^{p+1,q}, \quad d : C^{p,q} \to C^{p,q+1}$$

of module homomorphisms, defined for all integers p and q and such that

3.2.2. $$\delta\delta = 0, \quad \delta d = d\delta, \quad dd = 0.$$

Sometimes we call δ the horizontal differential and d the vertical differential.

A homomorphism $f : C_1 \to C_2$ of double complexes is a homomorphism of bigraded modules, of degree 0 , with $f\delta = \delta f$ and $fd = df$.

Let X be a topological space, let $\mathfrak{U} = \{U_j; j \in J\}$ be an open covering of X and define

$$U_{j_0 \dots j_p} = \bigcap_{i=0,\dots,p} U_{j_i}$$

for $j_i \in J$.

Definition 3.2.3. Let $\mathfrak{U} = \{U_i; i \in J\}$ be an open covering of a space X and $\mathcal{L}^\bullet$ a differential graded presheaf of R-modules on X. Then we can construct a double complex $C^\bullet(\mathfrak{U}; \mathcal{L}^\bullet)$ of *Čech cochains* with coefficients in the differential graded presheaf $\mathcal{L}^\bullet$ by:

$$C^p(\mathfrak{U}; \mathcal{L}^q) = \prod_{(i_0,\dots,i_p) \in J^{p+1}} \mathcal{L}^q(U_{i_0 \dots i_p}).$$

For $\gamma \in C^p(\mathfrak{U}; \mathcal{L}^q)$ we denote its component in $\mathcal{L}^q(U_{i_0 \dots i_p})$ by $\gamma(U_{i_0 \dots i_p})$ to make its dependence on the open covering clear. The differentials are

$$\delta : C^p(\mathfrak{U}; \mathcal{L}^q) \to C^{p+1}(\mathfrak{U}; \mathcal{L}^q)$$

defined by

$$(\delta\gamma)(U_{i_0 \dots i_{p+1}}) = \sum_{j=0}^{p+1} (-1)^j \gamma(U_{i_0 \dots \hat{i}_j \dots i_{p+1}}) |_{U_{i_0 \dots i_{p+1}}}$$

for $\gamma \in C^p(\mathfrak{U}; \mathcal{L}^q)$ and

$$d : C^p(\mathfrak{U}; \mathcal{L}^q) \to C^p(\mathfrak{U}; \mathcal{L}^{q+1})$$

determined by $d : \mathcal{L}^q \to \mathcal{L}^{q+1}$. The subcomplex

$$C^p(\mathfrak{U};\mathcal{L}^q) = \{\gamma \in C^p(\mathfrak{U};\mathcal{L}^q) ; \gamma(U_{i_{\epsilon(0)}\dots i_{\epsilon(p)}}) = \operatorname{sign}(\epsilon)\gamma(U_{i_0\dots i_p})\}$$

where ϵ is a permutation of $\{0,\dots,p\}$, is called the subcomplex of *alternating cochains*.

We prove in Appendix A (Lemma A.1.2) that the inclusion

$$C^\bullet(\mathfrak{U};\mathcal{L}^\bullet) \longrightarrow C^\bullet(\mathfrak{U};\mathcal{L}^\bullet)$$

induces an isomorphism in cohomology.

Another way of describing alternating Čech cochains with coefficients in a (DG) presheaf is as follows. Choose an order in the covering $\mathfrak{U}$. Then

$$\check{C}^p(\mathfrak{U};\mathcal{L}^q) = \prod_{i_0<i_1<\dots<i_p} \mathcal{L}^q(U_{i_0\dots i_p}).$$

To the double complex $C^\bullet(\mathfrak{U};\mathcal{L}^\bullet)$ we can associate a single complex $T = \mathrm{STot}C^\bullet(\mathfrak{U};\mathcal{L}^\bullet)$, called the total complex[1] of the double complex, defined by

$$T^n = \bigoplus_{p+q=n} C^p(\mathfrak{U};\mathcal{L}^q)$$

and the differential $D : T^n \to T^{n+1}$ is defined by

$$D(x) = \delta(x) + (-1)^p d(x)$$

for $x \in C^p(\mathfrak{U};\mathcal{L}^q)$.

Definition 3.2.4. *Čech cohomology* of $\mathfrak{U}$ and $\mathcal{L}^\bullet$ is defined as

$$\check{H}^\bullet(\mathfrak{U};\mathcal{L}^\bullet) = \mathrm{H}(\mathrm{STot}(\check{C}^\bullet(\mathfrak{U};\mathcal{L}^\bullet))).$$

Let $f : X \to Y$ be a map, $\mathcal{L}^\bullet$ a DG presheaf on X, $\mathcal{M}^\bullet$ a DG presheaf on Y and $k^\bullet : \mathcal{M}^\bullet \to \mathcal{L}^\bullet$ a DG presheaf homomorphism over f. Then $k^\bullet$ induces a morphism of double complexes

$$k^{\bullet,\bullet} : C^\bullet(\mathfrak{U};\mathcal{M}^\bullet) \longrightarrow C^\bullet(f^{-1}\mathfrak{U};\mathcal{L}^\bullet)$$

and a homomorphism

$$k^\bullet : \check{H}^\bullet(\mathfrak{U};\mathcal{M}^\bullet) \longrightarrow \check{H}^\bullet(f^{-1}\mathfrak{U};\mathcal{L}^\bullet).$$

Let us return to the DG presheaf $\mathcal{L}^\bullet$ on X. Let $\mathfrak{V}$ be a refinement of the covering $\mathfrak{U}$. In this case we can choose a function, called projection, $\pi : \mathfrak{V} \to \mathfrak{U}$

[1] We use the unusual notation STot for the totalisation operator because we shall introduce another totalisation operator later on.

such that $V \subset \pi(V)$ for every $V \in \mathfrak{V}$. Such a projection induces a morphism of double complexes

$$\pi : C^{\cdot}(\mathfrak{U}; \mathcal{L}^{\bullet}) \to C^{\cdot}(\mathfrak{V}; \mathcal{L}^{\bullet})$$

defined by

$$(\pi\gamma)(V_{i_0 \ldots i_p}) = \gamma(\pi(V_{i_0}) \cap \ldots \cap \pi(V_{i_p})) \,|_{V_{i_0 \ldots i_p}}$$

for $\gamma \in C^p(\mathfrak{U}; \mathcal{L}^{\bullet})$ and $V_{i_j} \in \mathfrak{V}$. We shall show in Appendix A (Lemma A.1.1) that any two projections from $\mathfrak{V}$ to $\mathfrak{U}$ induce cochain homotopic morphisms of the associated total complexes.

Therefore for every refinement $\mathfrak{V}$ of $\mathfrak{U}$ there is a well-defined morphism in cohomology, independent of the projections:

$$\check{H}^{\cdot}(\mathfrak{U}; \mathcal{L}^{\bullet}) \to \check{H}^{\cdot}(\mathfrak{V}; \mathcal{L}^{\bullet}).$$

The relation being a refinement makes the set of open coverings of X into a direct system, therefore we can define

$$\check{H}^{\bullet}(X; \mathcal{L}^{\bullet}) = \varinjlim \check{H}^{\cdot}(\mathfrak{U}; \mathcal{L}^{\bullet}).$$

Let $k^{\bullet} : \mathcal{M}^{\bullet} \to \mathcal{L}^{\bullet}$ be a DG presheaf homomorphism over $f : X \to Y$. Let $\mathfrak{U}$ and $\mathfrak{V}$ be open coverings of Y such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ and let $\pi : \mathfrak{V} \to \mathfrak{U}$ be a projection. Then there is a commutative diagram

$$\begin{array}{ccc} C^{\cdot}(\mathfrak{U}; \mathcal{M}^{\bullet}) & \xrightarrow{\pi} & C^{\cdot}(\mathfrak{V}; \mathcal{M}^{\bullet}) \\ \downarrow & & \downarrow \\ C^{\cdot}(f^{-1}\mathfrak{U}; \mathcal{L}^{\bullet}) & \xrightarrow{\pi} & C^{\cdot}(f^{-1}\mathfrak{V}; \mathcal{L}^{\bullet}) \end{array}$$

where the vertical arrows are the double complex homomorphisms $k^{\cdot\bullet}$. Thus we obtain a homomorphism

$$k^{\cdot} : \check{H}^{\cdot}(Y; \mathcal{M}^{\bullet}) \longrightarrow \varinjlim_{\mathfrak{U}} \check{H}^{\cdot}(f^{-1}\mathfrak{U}; \mathcal{L}^{\bullet}) \longrightarrow \check{H}^{\cdot}(X; \mathcal{L}^{\bullet})$$

where the direct limit is over the open coverings $\mathfrak{U}$ of Y.

Let us mention another property of Čech cohomology, the exact sequence of a pair.

Let X be a space and A a closed subset in X. If $\mathcal{L}^{\bullet}$ is a DG presheaf on X, we can define the following DG presheaves.

$$\mathcal{L}^{\bullet}_{(X,A)}(U) = \begin{cases} \mathcal{L}^{\bullet}(U) & \text{if } U \cap A = \emptyset \\ 0 & \text{if } U \cap A \neq \emptyset \end{cases}$$

$$\mathcal{L}^{\bullet}_{X|A}(U) = \begin{cases} \mathcal{L}^{\bullet}(U) & \text{if } U \cap A \neq \emptyset \\ 0 & \text{if } U \cap A = \emptyset \end{cases}$$

These DG presheaves form a short exact sequence

$$0 \to \mathcal{L}^\bullet_{(X,A)} \longrightarrow \mathcal{L}^\bullet \longrightarrow \mathcal{L}^\bullet_{X|A} \to 0\,.$$

Therefore they induce a long exact sequence in Čech cohomology as we show by the next proposition.

Proposition 3.2.5. *For every short exact sequence*

$$0 \to \mathcal{L}^\bullet \to \mathcal{M}^\bullet \to \mathcal{Q}^\bullet \to 0$$

of DG presheaves on a space X there exists the associated long exact sequence of $\check{H}^*(X;-)$.

Proof of 3.2.5: For every open set U in X the sequence of complexes

$$0 \to \mathcal{L}^\bullet(U) \to \mathcal{M}^\bullet(U) \to \mathcal{Q}^\bullet(U) \to 0$$

is by definition exact. Therefore, for every open covering $\mathfrak{U}$, we obtain a short exact sequence

$$0 \to C^*(\mathfrak{U};\mathcal{L}^\bullet) \to C^*(\mathfrak{U};\mathcal{M}^\bullet) \to C^*(\mathfrak{U};\mathcal{Q}^\bullet) \to 0$$

and a short exact sequence of total complexes. Applying the homology functor gives the long exact sequence of $\check{H}^*(\mathfrak{U};-)$. With the direct limit process we obtain the long exact sequence of $\check{H}^*(X;-)$. $\boxed{3.2.5}$

3.3. Localised equivariant cohomology on compacta

In this section we show that localised equivariant cohomology of nice compact $\mathbf{T}$-pairs can be expressed as Čech cohomology of certain DG presheaves.

The generalised Mayer-Vietoris exact sequence of singular cochains is usually described as follows. Let X be a space, $\mathfrak{U} = \{U_\alpha; \alpha \in J\}$ an open covering of X. Singular simplexes which lie in a member of the covering $\mathfrak{U}$ are called $\mathfrak{U}$-small. We denote by $S^*_{\mathfrak{U}}(X)$ the cochain complex of those singular cochains which are zero on all singular simplexes which are not $\mathfrak{U}$-small. Let $\epsilon : S^*_{\mathfrak{U}}(X) \to \prod_{\alpha\in J} S^*(U_\alpha)$ be the restriction homomorphism

$$\gamma \mapsto \sum_{\alpha} \gamma|_{U_{\alpha}}\,.$$

Choose an order in the indexing family J of the covering $\mathfrak{U}$. Then there is an exact sequence

$$0 \to S^*_{\mathfrak{U}}(X) \xrightarrow{\epsilon} \prod_{\alpha_0\in J} S^*(U_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} S^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1<\alpha_2} S^*(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots$$

where δ is the Čech differential. This holds for singular cochains with any ring of coefficients.

Assume now that X is a $\mathbf{T}$-space and that $\mathfrak{U}$ is a finite, invariant, ordered covering of X with n elements. Because $\mathbf{Z}[u,u^{-1}]$ is a free $\mathbf{Z}$-module and $\mathfrak{U}$ is finite, the following sequence is also exact.

$$0 \to S^*_{\mathfrak{U}}(X) \otimes \mathbf{Z}[u,u^{-1}] \xrightarrow{\epsilon} \bigoplus_{i \in I} S^*(U_i) \otimes \mathbf{Z}[u,u^{-1}] \xrightarrow{\delta}$$

3.3.1.

$$\bigoplus_{i<j} S^*(U_{i,j}) \otimes \mathbf{Z}[u,u^{-1}] \xrightarrow{\delta} \cdots \xrightarrow{\delta} S^*(U_{1,2,\ldots,n}) \otimes \mathbf{Z}[u,u^{-1}] \to 0$$

We can reformulate this in terms of alternating Čech cochains with coefficients in DG presheaves as follows.

The functor

$$U \mapsto S^*(U)[u,u^{-1}]$$

for U open invariant subsets of a $\mathbf{T}$-space X is a DG presheaf on the orbit space $X/\mathbf{T}$. However, as it intrinsically depends on X, we prefer to think of it as an object on X and call it *equivariant DG presheaf* on X. Similarly we shall call DG sheaves on $X/\mathbf{T}$, all of which will depend on X, equivariant DG sheaves on X. Note that if the $\mathbf{T}$-space X is metrisable, so is the orbit space $X/\mathbf{T}$.

Definition 3.3.2. Let X be a $\mathbf{T}$-space and let ∂^* be the differential in the DG presheaf $S^*(-)$. We define an equivariant DG presheaf $(\mathcal{P}^\bullet, d)$ on X by

$$\mathcal{P}^\bullet(U) = S^*(U) \otimes \mathbf{Z}[u,u^{-1}]$$

for any invariant open subset U of X, with the differential $d = \partial^* + \mathrm{I}u$.

Notice that this DG presheaf is not bounded below. When we need to indicate the underlying space of this equivariant DG presheaf we shall write it as $\mathcal{P}^\bullet_X$.

Our first aim in this section is to prove the following theorem.

Theorem 3.3.3. *There is an isomorphism*

$$\epsilon : u^{-1}H^*_{\mathbf{T}}(X;\mathbf{Z}) \xrightarrow{\cong} \check{H}^*(X;\mathcal{P}^\bullet)$$

for every compact $\mathbf{T}$*-space* X *and these isomorphisms are natural with respect to equivariant maps.*

As the operator I also commutes with the Čech differential we can write the exact sequence 3.3.1 as

$$0 \to S^*_{\mathfrak{U}}(X)[u,u^{-1}] \xrightarrow{\epsilon} \check{C}^0(\mathfrak{U};\mathcal{P}^\bullet) \xrightarrow{\delta} C^1(\mathfrak{U};\mathcal{P}^\bullet) \xrightarrow{\delta} C^2(\mathfrak{U};\mathcal{P}^\bullet) \xrightarrow{\delta} \cdots \quad \mathbf{3.3.4.}$$

Lemma 3.3.5. *There is a canonical isomorphism*

$$\epsilon : u^{-1} H^*_{\mathbb{T}}(X; \mathbb{Z}) \cong \check{H}^*(\mathfrak{U}; \mathcal{P}^\bullet)$$

for any $\mathbb{T}$*-space* X *and finite invariant open covering* $\mathfrak{U}$ *of* X.

Proof of 3.3.5: Since $\mathfrak{U}$ is finite, the exact sequence 3.3.4 implies that the differential δ in the double complex $\check{C}^*(\mathfrak{U}; \mathcal{P}^\bullet)$ is exact. Therefore, by a standard double complex argument, every element $[\xi]$ of degree p in $\mathrm{H}(\mathrm{STot}\check{C}^*(\mathfrak{U}; \mathcal{P}^\bullet))$ can be represented by an element $\xi \in C^0(\mathfrak{U}; \mathcal{P}^p)$ such that $d\xi = 0$ and $\delta\xi = 0$. Because of the latter $\xi = \epsilon(\xi')$ for some element $\xi' \in (S^*_{\mathfrak{U}}(X)[u, u^{-1}])^p$ such that $d\xi' = 0$. Thus

$$\epsilon : \mathrm{H}(S^*_{\mathfrak{U}}(X)[u, u^{-1}]) \longrightarrow \check{H}^*(\mathfrak{U}; \mathcal{P}^\bullet)$$

is surjective. To show that ϵ is also injective note that if $\epsilon(\eta) = (d + \delta)\phi$ for a cocycle η in $S^*_{\mathfrak{U}}(X)[u, u^{-1}]$ and an element $\phi \in C^0(\mathfrak{U}; \mathcal{P}^\bullet)$, then $\delta\phi = 0$. Therefore $\phi = \epsilon(\phi')$ for some $\phi' \in S^*_{\mathfrak{U}}(X)[u, u^{-1}]$ and as $\epsilon(d\phi') = \epsilon(\eta)$ and ϵ is injective, we conclude $\eta = d\phi'$. To show that the inclusion

$$S^*_{\mathfrak{U}}(X)[u, u^{-1}] \hookrightarrow S^*(X)[u, u^{-1}]$$

induces an isomorphism in cohomology consider the exact sequence

$$0 \to S^*_{\mathfrak{U}}(X) \hookrightarrow S^*(X) \to Q^* \to 0.$$

As the inclusion $S^*_{\mathfrak{U}}(X) \hookrightarrow S^*(X)$ induces an isomorphism in cohomology, the cohomology of Q^* is trivial.By the same double complex argument also the cohomology of $(Q^*[u, u^{-1}], d_Q + \mathrm{I}u)$ is trivial. The exact sequence

$$0 \to S^*_{\mathfrak{U}}(X)[u, u^{-1}] \hookrightarrow S^*(X)[u, u^{-1}] \to Q^*[u, u^{-1}] \to 0$$

then implies that the inclusion induces an isomorphism. Thus also the natural homomorphism $\epsilon : S^*(X)[u, u^{-1}] \longrightarrow \mathrm{STot}C^*(\mathfrak{U}; \mathcal{P}^\bullet)$ induces an isomorphism $\mathrm{H}(S^*(X)[u, u^{-1}]) \cong \check{H}^*(\mathfrak{U}; \mathcal{P}^\bullet)$. By [Jones 87, §5] we obtain the desired isomorphism. $\boxed{3.3.5}$

Proof of 3.3.3: Let X be a compact $\mathbb{T}$-space and let $\mathfrak{V}$ be a finite, invariant open refinement of the covering $\mathfrak{U}$ of X. Every projection ϕ induces a homomorphism of the exact sequences

$$\begin{array}{ccccccc}
S^*(X)[u, u^{-1}] & \xrightarrow{\epsilon} & \check{C}^0(\mathfrak{U}; \mathcal{P}^\bullet) & \xrightarrow{\delta} & C^1(\mathfrak{U}; \mathcal{P}^\bullet) & \xrightarrow{\delta} & \cdots \\
\| & & \downarrow & & \downarrow & & \\
S^*(X)[u, u^{-1}] & \xrightarrow{\epsilon} & C^0(\mathfrak{V}; \mathcal{P}^\bullet) & \xrightarrow{\delta} & C^1(\mathfrak{V}; \mathcal{P}^\bullet) & \xrightarrow{\delta} & \cdots .
\end{array}$$

We can write this as a homomorphism of double complexes

$$\check{C}^*(\mathfrak{U}; \mathcal{P}^\bullet) \longrightarrow \check{C}^*(\mathfrak{V}; \mathcal{P}^\bullet)$$

which makes the following diagram commutative.

$$\begin{tikzcd} & u^{-1}H_{\mathbb{T}}^*(X) \arrow[dl, "\epsilon"'] \arrow[dr, "\epsilon"] & \\ H(\mathfrak{U};\mathcal{P}^\bullet) \arrow[rr, "\cong"] & & H(\mathfrak{V};\mathcal{P}^\bullet) \end{tikzcd}$$

Since the finite invariant coverings are cofinal in the category of all invariant open coverings, we obtain an isomorphism

$$\epsilon : u^{-1}H_{\mathbb{T}}^*(X) \cong \check{H}^*(X;\mathcal{P}^\bullet)\,.$$

To show that these isomorphisms are natural let us first consider how equivariant maps induce homomorphisms in Čech cohomology.

Let $f : X \to Y$ be an equivariant map between compact $\mathbb{T}$-spaces X and Y. The pull-back of singular cochains induces an equivariant DG presheaf homomorphism $f^\bullet : \mathcal{P}^\bullet_Y \to \mathcal{P}^\bullet_X$. We obtain a morphism of exact sequences

$$\begin{tikzcd} S^\bullet(Y)[u,u^{-1}] \arrow[r, "\iota"] \arrow[d] & \check{C}^0(\mathfrak{U};\mathcal{P}^\bullet_Y) \arrow[r, "\delta"] \arrow[d] & C^1(\mathfrak{U};\mathcal{P}^\bullet_Y) \arrow[r, "\delta"] \arrow[d] & \cdots \\ S^\bullet(X)[u,u^{-1}] \arrow[r, "\iota"] & C^0(f^{-1}\mathfrak{U};\mathcal{P}^\bullet_X) \arrow[r, "\delta"] & C^1(f^{-1}\mathfrak{U};\mathcal{P}^\bullet_X) \arrow[r, "\delta"] & \cdots \end{tikzcd}$$

for any finite invariant open covering $\mathfrak{U}$ of Y.

This homomorphism induces the following commutative diagram.

$$\begin{tikzcd} u^{-1}H_{\mathbb{T}}^*(Y) \arrow[r, phantom, "\cong"] \arrow[d] & \check{H}^*(\mathfrak{U};\mathcal{P}^\bullet_Y) \arrow[d] \\ u^{-1}H_{\mathbb{T}}^*(X) \arrow[r, phantom, "\cong"] & \check{H}^*(f^{-1}\mathfrak{U};\mathcal{P}^\bullet_X) \end{tikzcd}$$

These diagrams are compatible with projections to refinements of the invariant open coverings and because X and Y are compact, we obtain a commutative diagram

$$\begin{tikzcd} u^{-1}H_{\mathbb{T}}^*(Y) \arrow[r, phantom, "\cong"] \arrow[d] & \check{H}^*(Y;\mathcal{P}^\bullet_Y) \arrow[d] \\ u^{-1}H_{\mathbb{T}}^*(X) \arrow[r, phantom, "\cong"] & \check{H}^*(X;\mathcal{P}^\bullet_X). \end{tikzcd}$$

$\boxed{3.3.3}$

Now we extend this isomorphism of functors to nice $\mathbb{T}$-pairs.

Definition 3.3.6. A $\mathbb{T}$-pair (X, A) with A a closed subspace of X is called *admissible* if it has the following property. If $\mathfrak{U}$ is a family of invariant open sets in X which covers A, then there exists an invariant open neighbourhood E of A and an equivariant deformation retraction

$$D_t : E \to E, \quad \text{for } t \in [0,1]$$

such that $D_0 = \mathrm{Id}_E$, $D_1 = r : E \to A$ and

- for every subset B in A the restriction $D_t|_{r^{-1}(B)}$ is an equivariant deformation retraction of $r^{-1}(B)$ onto B
- for every point $x \in A$ there exists an invariant open neighbourhood V_x of x in A with $r^{-1}(V_x) \subset U_x$ for some $U_x \in \mathfrak{U}$.

Example: Any smooth $\mathbf{T}$-pair (X, A) such that A has an invariant tubular neighbourhood in X, in particular every compact smooth $\mathbf{T}$-pair is admissible.

Definition 3.3.7. For a $\mathbf{T}$-pair (X, A) we define an equivariant DG presheaf $\mathcal{P}^\bullet_{(X,A)}$ on X by

$$\mathcal{P}^\bullet_{(X,A)}(U) = \begin{cases} S^*(U) \otimes \mathbb{Z}[u, u^{-1}] & \text{if } U \cap A = \emptyset \\ 0 & \text{if } U \cap A \neq \emptyset \end{cases}$$

for any invariant open subset U of X, and an equivariant differential graded presheaf $\mathcal{P}^\bullet_{X|A}$ by

$$\mathcal{P}^\bullet_{X|A}(U) = \begin{cases} S^*(U) \otimes \mathbb{Z}[u, u^{-1}] & \text{if } U \cap A \neq \emptyset \\ 0 & \text{if } U \cap A = \emptyset \end{cases}$$

for any invariant open subset U of X.

These equivariant DG presheaves are natural in the following sense. Every $\mathbf{T}$-map $f : (X, A) \to (Y, B)$ induces an equivariant DG presheaf homomorphism $f^\bullet : \mathcal{P}^\bullet_{(Y,B)} \to \mathcal{P}^\bullet_{(X,A)}$ over f, given by

$$f^\bullet_U : \mathcal{P}^\bullet_{(Y,B)}(U) \longrightarrow \mathcal{P}^\bullet_{(X,A)}(f^{-1}U), \quad \sum \phi_i u^i \mapsto \sum f^\sharp \phi_i u^i$$

where $f^\sharp \phi_i$ denotes the pull-back of the singular cochain ϕ_i by the map f. This assignment is functorial, i.e. $(gf)^\bullet = f^\bullet g^\bullet$.

We want to prove the following theorem.

Theorem 3.3.8. *On the category of compact admissible* $\mathbf{T}$*-pairs and* $\mathbf{T}$*-maps there is an isomorphism of functors*

$$\epsilon : u^{-1} H^*_{\mathbf{T}}(X, A) \xrightarrow{\cong} \mathbb{H}^*(X, A; \mathcal{P}^\bullet_{(X,A)}).$$

Proof of 3.3.8: Let us consider a compact admissible $\mathbf{T}$-pair (X, A) now. Let $\mathfrak{U}$ be a finite invariant open covering of X such that

$$E = \bigcup_{\substack{U \in \mathfrak{U} \\ U \cap A \neq \emptyset}} U$$

equivariantly deformation retracts onto A by an equivariant retraction r and

$$U \in \mathfrak{U}, U \cap A \neq \emptyset \Rightarrow U = r^{-1}(U \cap A).$$

Let us call such a covering $\mathfrak{U}$ of X an ideal covering of the compact admissible $\mathbf{T}$-pair (X, A) and let us call E the ideal neighbourhood of A associated to $\mathfrak{U}$. It is easy to see that every compact admissible $\mathbf{T}$-pair admits arbitrarily fine ideal coverings.

We have the following morphism of short exact sequences of cochain complexes

$$\begin{array}{ccccc} S^*(E)[u,u^{-1}] & \leftarrow & S^*(X)[u,u^{-1}] & \leftarrow & \mathrm{Hom}(S_*(X)/S_*(E);\mathbb{Z})[u,u^{-1}] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{STot}\,C^*(\mathfrak{U};\mathcal{P}^\bullet_{X|A}) & \leftarrow & \mathrm{STot}\,C^*(\mathfrak{U};\mathcal{P}^\bullet_X) & \leftarrow & \mathrm{STot}\,C^*(\mathfrak{U};\mathcal{P}^\bullet_{(X,A)}) \end{array}$$

where the vertical arrows have targets in the nontrivial components of the respective C^0. The middle vertical arrow induces an isomorphism in cohomology. To see that the induced homomorphisms

$$u^{-1}H^*_{\mathbf{T}}(E) \to \check{H}^*(\mathfrak{U};\mathcal{P}^\bullet_{X|A})$$

are isomorphisms define $\mathfrak{U}'$ to be the subset of those elements of $\mathfrak{U}$ which meet A nontrivially. Because $\mathfrak{U}'$ is a finite invariant open covering of E Lemma 3.3.5 implies that this homomorphism is the isomorphism

$$u^{-1}H^*_{\mathbf{T}}(E) \cong \check{H}^*(\mathfrak{U}';\mathcal{P}^\bullet_E) = \check{H}^*(\mathfrak{U};\mathcal{P}^\bullet_{X|A}).$$

By 5-lemma applied to the induced morphism of the long exact sequences we obtain an isomorphism

$$u^{-1}H^*_{\mathbf{T}}(X,E) \to \check{H}^*(\mathfrak{U};\mathcal{P}^\bullet_{(X,A)})$$

and by homotopy invariance the natural homomorphism

$$\epsilon : u^{-1}H^*_{\mathbf{T}}(X,A) \to \check{H}^*(\mathfrak{U};\mathcal{P}^\bullet_{(X,A)})$$

is an isomorphism.

Remark: Notice that if the ideal neighbourhood E in the short exact sequence above is replaced by a larger ideal neighbourhood E_1 of A, we obtain a similar morphism of short exact sequences

$$\begin{array}{ccccc} S^*(E_1)[u,u^{-1}] & \leftarrow & S^*(X)[u,u^{-1}] & \leftarrow & \mathrm{Hom}(S_*(X)/S_*(E_1);\mathbb{Z})[u,u^{-1}] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{STot}\,C^*(\mathfrak{U};\mathcal{P}^\bullet_{X|A}) & \leftarrow & \mathrm{STot}\,C^*(\mathfrak{U};\mathcal{P}^\bullet_X) & \leftarrow & \mathrm{STot}\,C^*(\mathfrak{U};\mathcal{P}^\bullet_{(X,A)}). \end{array}$$

This short exact sequence induces an isomorphism of long exact sequences and the inclusion $(X, E) \hookrightarrow (X, E_1)$ induces a morphism of commutative diagrams as above.

Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be ideal coverings of (X, A) such that $\mathfrak{U}_2$ is a refinement of $\mathfrak{U}_1$. Let E_1 and E_2 be the ideal neighbourhoods of A associated to $\mathfrak{U}_1$ and $\mathfrak{U}_2$, respectively. Then it follows from the remark above that we obtain a commutative diagram

$$\begin{array}{ccc} S^*(X, E_1)[u, u^{-1}] & \longrightarrow & S^*(X, E_2)[u, u^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{STot}C^*(\mathfrak{U}_1; \mathcal{P}^\bullet_{(X,A)}) & \longrightarrow & \mathrm{STot}C^*(\mathfrak{U}_2; \mathcal{P}^\bullet_{(X,A)}). \end{array}$$

Both vertical arrows induce isomorphisms in cohomology and so does the upper horizontal one. Therefore the projection homomorphisms on such coverings induce isomorphisms in Čech cohomology. We define

$$\check{H}^*(X, A; \mathcal{P}^\bullet_{(X,A)}) = \varinjlim H^*(\mathfrak{U}, \mathcal{P}^\bullet_{(X,A)})$$

where the limit is over all invariant open coverings of X.

Then the natural homomorphisms ϵ induce an isomorphism

$$\epsilon : u^{-1}H^*_{\mathbf{T}}(X, A) \xrightarrow{\cong} \check{H}^*(X, A; \mathcal{P}^\bullet_{(X,A)}).$$

Let $f : (X, A) \to (Y, B)$ be a $\mathbf{T}$-map between two compact admissible $\mathbf{T}$-pairs. Let $\mathfrak{U}$ be an ideal covering of the pair (Y, B) and E the ideal neighbourhood of B associated to $\mathfrak{U}$. Then we can represent $u^{-1}H^*_{\mathbf{T}}(Y, B)$ by $S^*(Y, E)[u, u^{-1}]$. If $\mathfrak{V}$ is an ideal covering of (X, A) such that $\mathfrak{V}$ is a refinement of $f^{-1}\mathfrak{U}$, then the following diagram commutes

$$\begin{tikzcd} S^*(Y, E)[u, u^{-1}] \arrow[r, "f^*"] \arrow[d] & S^*(X, A)[u, u^{-1}] \arrow[dr] & \\ C^0(\mathfrak{U}; \mathcal{P}^\bullet_{(Y,B)}) \arrow[r, "f^\bullet"] & C^0(f^{-1}\mathfrak{U}; \mathcal{P}^\bullet_{(X,A)}) \arrow[r] & C^0(\mathfrak{V}; \mathcal{P}^\bullet_{(X,A)}) \end{tikzcd}$$

where the downwards pointing maps are the natural homomorphisms ϵ. As the two homomorphisms ϵ induce isomorphisms into the direct limits, this implies that the cohomology homomorphisms induced by $f^\bullet$ and f^* commute with the isomorphisms ϵ and thus the isomorphisms ϵ determine an isomorphism of functors. $\boxed{3.3.8}$

3.4. Jones-Petrack cohomology

In this section we discuss Jones-Petrack cohomology with integer coefficients. As we have shown in Section 1.5 localised equivariant cohomology does not satisfy Milnor's additivity property. Therefore we want to construct an equivariant cohomology, called Jones-Petrack cohomology, which

- agrees with localised equivariant cohomology on compact admissible T-pairs with finite cohomological dimension
- satisfies Milnor's additivity property.

Because the equivariant DG sheaves with which we expressed localised equivariant cohomology on nice compact pairs are not bounded below, Čech DG sheaf cohomology $\check{H}^\bullet(-;\mathcal{L}^\bullet)$ on infinite dimensional spaces does not have the properties we need. Therefore we use the completed cohomology $\check{H}^\bullet(X;\mathcal{L}^\bullet)$, of DG sheaves, which is constructed in Chapter 4 and satisfies the following axioms.

List of axioms **3.4.1.**

(a) $\check{H}^\bullet(X;\mathcal{L}^\bullet)$ is functorial with respect to DG sheaf homomorphisms over maps and there is a natural map

$$\kappa : \check{H}^\bullet(X;\mathcal{L}^\bullet) \longrightarrow \check{H}^\bullet(X;\mathcal{L}^\bullet).$$

(b) On every compact space with finite cohomological dimension [2] the natural map κ is an isomorphism.

(c) For every closed subset A of X and a DG sheaf $\mathcal{L}^\bullet_X$ on X there is a long exact sequence of a pair as follows.

$$\cdots \longrightarrow \check{H}^\bullet(X;\mathcal{L}^\bullet_{(X,A)}) \longrightarrow \check{H}^\bullet(X;\mathcal{L}_X{}^\bullet) \longrightarrow \check{H}^\bullet(X;\mathcal{L}^\bullet_{X|A}) \longrightarrow \check{H}^{\bullet+1}(X;\mathcal{L}^\bullet_{(X,A)}) \longrightarrow \cdots$$

These exact sequences are functorial.

The restriction to the closed subspace A induces an isomorphism

$$\check{H}^\bullet(X;\mathcal{L}^\bullet_{X|A}) \cong \check{H}^\bullet(A;\mathcal{L}^\bullet_{X|A}).$$

(d) Let $\phi : \mathcal{L}^\bullet \to \mathcal{M}^\bullet$ be a quasi-isomorphism of DG sheaves on X, i.e. a homomorphism which induces an isomorphism of the induced cohomology DG sheaves. Then ϕ induces an isomorphism in the completed cohomology

$$\check{H}^\bullet(X;\mathcal{L}^\bullet) \cong \check{H}^\bullet(X;\mathcal{M}^\bullet).$$

[2] The condition that the spaces be compact is in fact unnecessary, but we only need and prove the assertion in this case.

(e) Let $\mathcal{L}^\bullet$ be a DG sheaf on X. Then the inclusions

$$i_t : X \longrightarrow X \times I\,, \quad i_t(x) = (x, t)$$

for $t = 0, 1 \in I$ induce the same homomorphism

$$\check{H}^*(X \times I; \mathcal{L}^\bullet \times I) \longrightarrow \check{H}^*(X; \mathcal{L}^\bullet)$$

where $\mathcal{L}^\bullet \times I$ is the pull-back DG sheaf on $X \times I$ induced by the projection $X \times I \to X$.

(f) Let $\mathcal{L}^\bullet$ be a DG sheaf on X and let U be an open subset of X such that the associated cohomology DG sheaf $\mathcal{H}^\bullet$ of $\mathcal{L}^\bullet$ is trivial on U. Then the inclusion of $X - U$ in X induces an isomorphism

$$\check{H}^*(X; \mathcal{L}^\bullet) \cong \check{H}^*(X - U; \mathcal{L}^\bullet|_{X-U})\,.$$

(g) Let X_λ be a family of spaces and let $\mathcal{L}^\bullet_\lambda$ be a DG sheaf on X_λ. Then the inclusions of X_λ into their disjoint union induces an isomorphism

$$\check{H}^*(\coprod_\lambda X_\lambda; \coprod_\lambda \mathcal{L}^\bullet_\lambda) \cong \prod_\lambda \check{H}^*(X_\lambda; \mathcal{L}^\bullet_\lambda)\,.$$

Remark: Čech cohomology of DG sheaves satisfies all these axioms except for 3.4.1(d), 3.4.1(f) and 3.4.1(g), which need not be true on spaces with infinite cohomological dimension. One could prove this directly using [Godement 58]. We shall not do this, however, as this will follow from the facts that the completed cohomology satisfies these properties (Chapter 4) and that it agrees with Čech cohomology of DG sheaves on spaces with finite cohomological dimension (we will show this in Theorem A.3.1 for compact finite dimensional spaces only).

Let (X, A) be a T-pair and let $\mathcal{P}^\bullet_{(X,A)}$ be the equivariant DG presheaf as defined in Definition 3.3.7. Let $\mathcal{S}^\bullet_{(X,A)}$ be the equivariant DG sheaf associated to $\mathcal{P}^\bullet_{(X,A)}$. Then *Jones-Petrack cohomology* of (X, A) with coefficients in $\mathbf{Z}$ is

$$h^*_{\mathbf{T}}(X, A; \mathbf{Z}) = \check{H}(X; \mathcal{S}^\bullet_{(X,A)})\,.$$

To every equivariant map $f : (X, A) \to (Y, B)$ we assign the equivariant DG sheaf homomorphism $f^\bullet : \mathcal{S}^\bullet_{(Y,B)} \to \mathcal{S}^\bullet_{(X,A)}$ over f induced by the pull-back equivariant DG presheaf homomorphism $f^\bullet : \mathcal{P}^\bullet_{(Y,B)} \to \mathcal{P}^\bullet_{(X,A)}$ over f. Because of Axiom 3.4.1(a) this determines a functor from the category of admissible $\mathbf{T}$-pairs into the category of graded $\mathbf{Z}$-modules.

For a ring R and an admissible $\mathbf{T}$-pair (X, A) let $\mathcal{S}^\bullet_{(X,A;R)}$ to be the equivariant DG sheaf associated to the equivariant DG presheaf on X defined by

$$\mathcal{P}^\bullet_{(X,A;R)}(U) = \begin{cases} S^*(U; R) \otimes \mathbf{Z}[u, u^{-1}] & \text{if } U \cap A = \emptyset \\ 0 & \text{if } U \cap A \neq \emptyset \end{cases}$$

for any invariant open subset U of X. Define

$$h^*_{\mathbb{T}}(X, A; R) = \check{H}(X; \mathcal{S}^\bullet_{(X,A;R)}) .$$

Lemma 3.4.2. *There is an isomorphism of functors*

$$u^{-1}H^*_{\mathbb{T}}(X, A) \cong h^*_{\mathbb{T}}(X, A)$$

on the category of compact admissible $\mathbb{T}$*-pairs with finite cohomological dimension and equivariant maps.*

Proof of 3.4.2: In Appendix A we prove that

$$\check{H}^*(X; \mathcal{P}^\bullet_{(X,A)}) = \check{H}^*(X; \mathcal{S}^\bullet_{(X,A)}) .$$

Therefore this lemma follows directly from Theorem 3.3.8 and Axiom 3.4.1(b).
$\boxed{3.4.2}$

Proposition 3.4.3. *Let (X, A) be an admissible* $\mathbb{T}$*-pair. The inclusion $i : A \hookrightarrow X$ induces a homomorphism $i^\bullet : \mathcal{S}^\bullet_X \to \mathcal{S}^\bullet_A$ of equivariant DG sheaves over i which can be factorised as follows*

$$\begin{tikzcd} \mathcal{S}^\bullet_X \arrow[rr, "i^\bullet"] \arrow[dr] & & \mathcal{S}^\bullet_A \\ & \mathcal{S}^\bullet_{X|A} \arrow[ur, "\iota^\bullet"'] & \end{tikzcd}$$

where the morphism $\mathcal{S}^\bullet_X \to \mathcal{S}^\bullet_{X|A}$ is the natural projection and the morphism $\iota^\bullet$ is a quasi-isomorphism (if we regard $\mathcal{S}^\bullet_{X|A}$ as a DG sheaf on A).

Proof of 3.4.3: Let us show that $\iota^\bullet$ is a quasi-isomorphism. The equivariant DG sheaf homomorphism $i^\bullet$ over i is induced by the equivariant DG presheaf homomorphism over i given by

$$incl^\bullet_U : S^*(U)[u, u^{-1}] \to S^*(U \cap A)[u, u^{-1}] .$$

If the pair (X, A) is admissible, every point $a \in A$ has arbitrarily small invariant neighbourhoods U in X such that $incl^\bullet$ induces an isomorphism in cohomology and therefore the induced homomorphism

$$\mathrm{H}(i^\bullet_a) = \mathrm{H}(\iota^\bullet_a) : \mathrm{H}((\mathcal{S}^\bullet_{X|A})_a) \longrightarrow \mathrm{H}((\mathcal{S}^\bullet_A)_a)$$

is an isomorphism. $\boxed{3.4.3}$

Theorem 3.4.4. *The functor $h^*_{\mathbb{T}}(X, A)$ on the category of admissible* $\mathbb{T}$*-pairs is an equivariant cohomology theory satisfying Milnor's additivity axiom.*

Proof of 3.4.4: For the exact sequence of a pair observe that by Axiom 3.4.1(c) of the completed DG sheaf cohomology we have a long exact sequence which is almost the one requested, the only difference being that we have $\hat{H}^*(A; \mathcal{S}^\bullet_{X|A})$ instead of $h^*_{\mathbb{T}}(A)$. But Proposition 3.4.3 and Axiom 3.4.1(d) of the completed DG sheaf cohomology imply that the natural restriction homomorphism induced by the inclusion $A \hookrightarrow X$ provides us with an isomorphism

$$\hat{H}^*(A; \mathcal{S}^\bullet_{X|A}) \xrightarrow{\cong} h^*_{\mathbb{T}}(A).$$

This isomorphism is natural with respect to the homomorphisms over maps: If $f : (X, A) \to (Y, B)$ is a map of $\mathbb{T}$-pairs the commutative diagrams

$$\begin{array}{ccc} S^*(U)[u, u^{-1}] & \xrightarrow{f^*} & S^*(f^{-1}(U))[u, u^{-1}] \\ i^* \downarrow & & i^* \downarrow \\ S^*(U \cap B)[u, u^{-1}] & \xrightarrow{(f|_A)^*} & S^*(f^{-1}(U) \cap A)[u, u^{-1}] \end{array}$$

imply that the following diagram of equivariant DG sheaf homomorphisms is commutative.

$$\begin{array}{ccc} \mathcal{S}^\bullet_{Y|B} & \xrightarrow{f^\bullet} & \mathcal{S}^\bullet_{X|A} \\ \iota^\bullet \downarrow & & \iota^\bullet \downarrow \\ \mathcal{S}^\bullet_B & \xrightarrow{(f|_A)^\bullet} & \mathcal{S}^\bullet_A \end{array}$$

Axiom 3.4.1(c) and the identification of $\hat{H}^*(A; \mathcal{S}^\bullet_{X|A})$ with $\hat{H}^*(A; \mathcal{S}^\bullet_A)$ via the isomorphism induced by the quasi-isomorphism $\iota^\bullet$ provide us with the long exact sequences of admissible $\mathbb{T}$-pairs and these long exact sequences are functorial with respect to maps of admissible $\mathbb{T}$-pairs.

For the equivariant homotopy invariance notice that the projection $\pi : X \times I \to X$ induces an equivariant DG sheaf homomorphism over π

$$\pi^\bullet : \mathcal{S}^\bullet_X \longrightarrow \mathcal{S}^\bullet_{X \times I}$$

(or, in the relative case $\pi^\bullet : \mathcal{S}^\bullet_{X|A} \to \mathcal{S}^\bullet_{X \times I|A \times I}$) which factorises through the pull-back DG sheaf $\mathcal{S}^\bullet_X \times I = \pi^\sharp(\mathcal{S}^\bullet_X)$. The induced inclusion of equivariant DG sheaves on $X \times I$

$$\iota^\bullet : \mathcal{S}^\bullet_X \times I \longrightarrow \mathcal{S}^\bullet_{X \times I}$$

is a quasi-isomorphism as the inclusion

$$\pi^* S^*(U)[u, u^{-1}] \hookrightarrow S^*(U \times I)[u, u^{-1}]$$

induces an isomorphism in cohomology for any invariant open subset U in X (and similarly in the relative case). Thus Axioms 3.4.1(d) and 3.4.1(e) imply that the projection π induces canonical isomorphisms

$$h^*_{\mathbb{T}}(X) = h^*_{\mathbb{T}}(X \times I), \quad h^*_{\mathbb{T}}(X, A) = h^*_{\mathbb{T}}(X \times I, A \times I).$$

The excision property holds by Axiom 3.4.1(f) and the fact that $\mathcal{S}^*_{X|A}$ is trivial outside A.

Milnor's additivity property is a direct consequence of Axiom 3.4.1(g).

$\boxed{3.4.4}$

3.5. The fixed point theorem

Jones-Petrack cohomology satisfies the following fixed point theorem.

Theorem 3.5.1. *Let X be a space with a given circle action such that (X, F) is an admissible pair, where $F = X^{\mathbb{T}}$ is the fixed point set. Let R be a ring such that if X has an orbit with $\mathbb{Z}/n$ as its isotropy subgroup, then $n^{-1} \in R$. Then the restriction homomorphism*

$$h^*_{\mathbb{T}}(X; R) \longrightarrow h^*_{\mathbb{T}}(F; R)$$

is an isomorphism.

Proof of 3.5.1: For every non-fixed point orbit in this space we have $H^n_{\mathbb{T}}(\mathbb{T}.x; R) = 0$ for $n \neq 0$. If U is an invariant neighbourhood which equivariantly retracts onto such an orbit, then

$$u^{-1}H^*_{\mathbb{T}}(U; R) = \mathrm{H}(S^*(U; R)[u, u^{-1}]) = 0.$$

As every orbit has arbitrarily fine invariant neighbourhoods which equivariantly retract onto it, we have $\mathcal{H}^*_x = 0$, for every non-fixed point x and the cohomology DG sheaf $\mathcal{H}^*$ of $\mathcal{S}^*$. Thus, by Axiom 3.4.1(f), we can excise the non-fixed points. By Proposition 3.4.3 we obtain the result. $\boxed{3.5.1}$

For nice spaces with trivial circle action we can calculate the Jones-Petrack cohomology explicitly.

Lemma 3.5.2. *Let X be a locally contractible space with the trivial circle action and let R be a ring. Then*

$$h^n_{\mathbb{T}}(X; R) \cong \prod_{i \in \mathbb{Z}} H^{n+2i}(X; R)\,.$$

Proof of 3.5.2: The DG sheaf homomorphism corresponding to the DG presheaf inclusion

$$R[u, u^{-1}] \longrightarrow S^*(-; R)[u, u^{-1}]$$

is a quasi-isomorphism and therefore induces an isomorphism in the completed DG sheaf cohomology. The completed DG sheaf cohomology of the constant DG sheaf $\underline{R}^{2i} = R, \underline{R}^{2i+1} = 0$ is the tensor product of the graded algebras

$$H^*(X; R) \otimes_R R[[u, u^{-1}]]\,.$$

$\boxed{3.5.2}$

This lemma and the fixed point theorem show that Jones-Petrack cohomology with coefficients in $\mathbf{C}$ is isomorphic to the completed periodic equivariant cohomology of [Jones, Petrack 90].

3.6. De Rham version

Let X be a smooth manifold (for the definiton of infinite dimensional smooth manifolds see [Beggs 87]) with a given action of the circle $\mathbf{T}$. We assume every orbit has a neighbourhood which smoothly and equivariantly retracts onto the orbit. The assignment

$$U \mapsto \Omega^*_{\mathrm{inv}}(U)[u, u^{-1}]$$

for every invariant open subset U of X, with the differential $d + \iota u$, where ι is the interior product with the vector field of the circle action, determines an equivariant DG presheaf. Notice that in case X is finite dimensional, this is in fact an equivariant DG sheaf. In case X is infinite dimensional, however, it does not satisfy the collation property and is therefore not a DG sheaf. In any case let $\mathcal{R}^\bullet$ denote the associated DG sheaf.

If $f : X \to Y$ is a smooth map of $\mathbf{T}$-manifolds, the pull-back of the differential forms induces a well-defined equivariant DG sheaf homomorphism $f^\bullet : \mathcal{R}^\bullet_Y \to \mathcal{R}^\bullet_X$ over f.

We define the de Rham version of the Jones-Petrack cohomology by

$$h^*_{\mathbf{T}}(X; \mathbf{C}) = H^*(X; \mathcal{R}^\bullet)$$

and the smooth $\mathbf{T}$-maps induce the corresponding homomorphisms via the induced homomorphisms of equivariant DG sheaves over them.

Theorem 3.6.1. *Let X be a smooth $\mathbf{T}$-manifold and let the fixed points form a smooth submanifold F such that F has a tubular neighbourhood in X. Then the inclusion $i : F \hookrightarrow X$ induces an isomorphism*

$$i^* : h^*_{\mathbf{T}}(X; \mathbf{C}) \longrightarrow h^*_{\mathbf{T}}(F; \mathbf{C}) .$$

Proof of 3.6.1: The homomorphism $i^\bullet$ over i can be factorised as follows

$$\begin{tikzcd}
\mathcal{R}^\bullet_X \arrow[rr, "i^\bullet"] \arrow[dr] & & \mathcal{R}^\bullet_F \\
& \mathcal{R}^\bullet_{X|F} \arrow[ur] &
\end{tikzcd}$$

where the homomorphism $\mathcal{R}^\bullet_X \to \mathcal{R}^\bullet_{X|F}$ is the natural projection. The assumption on the retracting neighbourhoods of orbits implies that every non-fixed point orbit x has arbitrarily fine invariant neighbourhoods with trivial localised equivariant cohomology and so the cohomology of the stalk $\mathrm{H}(\mathcal{R}^\bullet_{X,x})$ is trivial. Therefore $\hat{H}^*(X;\mathcal{R}^\bullet_X) \to H^*(F;\mathcal{R}^\bullet_{X|F})$ is an isomorphism by Axiom 3.4.1(f). The assumption that F has a tubular neighbourhood in X implies that $\mathcal{R}^\bullet_{X|F} \to \mathcal{R}^\bullet_F$ is a quasi-isomorphism (as in the proof of Proposition 3.4.3) and therefore induces an isomorphism in the completed DG sheaf cohomology. $\boxed{3.6.1}$

Chapter 4

Completed cohomology of DG sheaves

We have constructed the desired Jones-Petrack cohomology theory assuming we had a suitable cohomology theory of DG sheaves unbounded below. In particular we assumed the following properties.

- On compact finite dimensional spaces it agrees with Čech cohomology of DG sheaves.
- It satisfies suitable Eilenberg-Steenrod properties.
- A quasi-isomorphism, i.e. a homomorphism of DG sheaves which induces an isomorphism of the associated cohomology sheaves, induces an isomorphism in the cohomology.
- It satisfies Milnor's additivity axiom.

In case of DG sheaves which are bounded below Čech cohomology of DG sheaves satisfies all these properties. This need not be true for DG sheaves which are not bounded below if the underlying space is infinite dimensional. In this chapter we construct a cohomology theory, which we call completed cohomology, for DG sheaves, which has the desired properties. Its disadvantage is that on infinite dimensional spaces completed cohomology of DG sheaves with all components fine or flabby may not agree with the cohomology of global sections.

4.1. Unbounded double complexes

In section 3.2 we defined a double complex. Here we recollect some properties of double complexes needed in the sequel.

Let $C = (C^{p,q}; \delta, d)$ be a double complex. The second cohomology H_{II} of C is formed with respect to d in the usual way as

$$H_{II}^{p,q}(C) = \mathrm{H}(C^{p,q}; d) .$$

It is a bigraded object with a differential $\delta : H_{II}^{p,q} \to H_{II}^{p+1,q}$ induced by the original δ. In turn its cohomology

$$H_I^p H_{II}^q(C) = \mathrm{H}(H_{II}^{p,q}; \delta)$$

is a bigraded object. The first cohomology and the iterated cohomology $H_{II} H_I C$ are defined similarly.

Definition 4.1.1. There are two *totalisation operators* STot and PTot which transform a double complex into a single complex. If C is a double complex then $X = \mathrm{STot}(C)$ and $Y = \mathrm{PTot}(C)$ are defined by

$$X^n = \bigoplus_{p+q=n} C^{p,q}$$

with boundary map $D : X^n \to X^{n+1}$ defined by $D(x) = \delta(x) + (-1)^p d(x)$ for $x \in C^{p,q}$ and

$$Y^n = \prod_{p+q=n} C^{p,q} ,$$

with boundary map $D : Y^n \to Y^{n+1}$ defined by $D(x) = \delta(x) + (-1)^p d(x)$ for $x \in C^{p,q}$.

The commutativity of the differentials implies that $D^2 = 0$ in both X and Y.

There is a natural injection

$$\kappa : \mathrm{STot}(C) \longrightarrow \mathrm{PTot}(C).$$

We shall only deal with double complexes such that there exists an integer p_0 so that $C^{p,q} = 0$ for $p < p_0$. Even in this case the two totalisations in general do not agree. If there exists also an integer p_1 such that $C^{p,q} = 0$ for $p > p_1$ then the two totalisations agree.

The *first filtrations* F_I of $\mathrm{STot}(C)$ and $\mathrm{PTot}(C)$ are defined by the subcomplexes $F_{I,p}$ with

$$(F_{I,p}\mathrm{STot}(C))_n = \bigoplus_{h \le p} C^{h,n-h} \quad \textit{and} \quad (F_{I,p}\mathrm{PTot}(C))_n = \prod_{h \le p} C^{h,n-h} .$$

respectively. The associated spectral sequence of F_I is called the *first spectral sequence* E_I of the double complex [Godement 58]. As we shall always use only the first spectral sequence, we drop the index from now on. We begin the spectral sequence with $E_0^{p,q} = C^{p,q}$.

Lemma 4.1.2. *Let $C = \{C^{p,q}; \delta, d\}$ be a double complex such that $C^{p,q} = 0$ if $p < 0$. If $E_2(C) = 0$, then* $\mathrm{H}(\mathrm{PTot}(C)) = 0$.

Proof of 4.1.2: Let $\sum_i \gamma_i$, $\gamma_i \in C^{i,n-i}$ be an n-cocycle of the total complex $\mathrm{PTot}(C)$. Therefore

$$d\gamma_0 = 0\,, \quad \delta\gamma_i = (-1)^i d\gamma_{i+1}$$

Because of $E_2 = 0$, the d-cocycle $\gamma_0 \in C^{0,n}$ is also a d-coboundary, i.e. there exists an element $\alpha_0 \in C^{0,n-1}$ such that $d\alpha_0 = \gamma_0$. Adding $-D\alpha_0 = -(\delta + d)\alpha_0$ to $\sum \gamma_i$ gives a cocycle in $\mathrm{PTot}(C)$, cohomologous to $\sum \gamma_i$, with its $C^{0,n}$-component being trivial. We continue in a similar way:

$$d(\gamma_1 - \delta\alpha_0) = 0\,, \quad \delta(\gamma_1 - \delta\alpha_0) = -d\gamma_2$$

therefore $E_2 = 0$ implies that there exist elements $\beta_0 \in C^{0,n-1}$, $\alpha_1 \in C^{1,n-2}$ such that

$$d\beta_0 = 0\,, \quad \delta\beta_0 - d\alpha_1 = \gamma_1 - \delta\alpha_0$$

Adding $-D(\beta_0 + \alpha_1)$ to $-\delta\alpha_0 + \sum_{i\geq 1}\gamma_i$ gives the cocycle $-\delta\alpha_1 + \sum_{i\geq 2}\gamma_i$ which has no $C^{0,n}$ or $C^{1,n-1}$ component. If we continue to construct α_i and β_i in this way, we obtain $\sum_{i\geq 0}(\alpha_i + \beta_i)$ such that

$$D(\sum_{i\geq 0}(\alpha_i + \beta_i)) = \sum_{i\geq 0}\gamma_i\,.$$

$\boxed{4.1.2}$

In general this is not true if we replace PTot by STot, as we may not be able to obtain a finite sum $\sum(\alpha_i + \beta_i)$ with the above mentioned properties.

Lemma 4.1.3. *Let A and B be double complexes such that $A^{p,q}$ and $B^{p,q}$ are trivial whenever $p < 0$. Let $f : A \to B$ be an injective homomorphism of double complexes such that the induced homomorphism $E_1(A) \to E_1(B)$ is injective and the induced homomorphism $E_2(A) \to E_2(B)$ is bijective. Then f induces an isomorphism* $\mathrm{H}(\mathrm{PTot}(A)) \to \mathrm{H}(\mathrm{PTot}(B))$.

Proof of 4.1.3: We have a short exact sequence of double complexes

$$0 \to A \longrightarrow B \longrightarrow Q \to 0$$

inducing a short exact sequence of the associated total complexes. On the other hand the short exact sequence of double complexes induces for every integer p a long exact sequence

$$\cdots \longrightarrow A_1^{p,q} \longrightarrow B_1^{p,q} \longrightarrow Q_1^{p,q} \longrightarrow A_1^{p,q+1} \longrightarrow \cdots$$

where $A_1^{p,q} = E_1^{p,q}(A)$ and similarly for B and Q. Because $A_1^{p,q} \to B_1^{p,q}$ is injective for all p and q, the homomorphisms $Q_1^{p,q} \to A_1^{p,q+1}$ are all trivial and therefore all the homomorphisms $B_1^{p,q} \to Q_1^{p,q}$ are surjective. Thus the above long exact sequences form a short exact sequence

$$0 \to E_1(A) \longrightarrow E_1(B) \longrightarrow E_1(Q) \to 0$$

of cochain complexes with respect to δ, which induces a long exact sequence

$$\cdots \longrightarrow A_2^{p,q} \longrightarrow B_2^{p,q} \longrightarrow Q_2^{p,q} \longrightarrow A_2^{p+1,q} \longrightarrow B_2^{p+1,q} \longrightarrow \cdots.$$

As the homomorphisms $A_2^{p,q} \to B_2^{p,q}$ in this long exact sequence are bijective, all the homomorphisms $B_2^{p,q} \to Q_2^{p,q}$ and $Q_2^{p,q} \to A_2^{p+1,q}$ are trivial and therefore $E_2(Q) = 0$. By Lemma 4.1.2 $\mathrm{H}(\mathrm{PTot}(Q)) = 0$ and therefore f induces an isomorphism $\mathrm{H}(\mathrm{PTot}(A)) \cong \mathrm{H}(\mathrm{PTot}(B))$. $\boxed{4.1.3}$

4.2. Definitions of cohomology theories

In this section we define cohomology theories for DG sheaves which are not necessarily bounded below. There are essentially two ways in which we can define such cohomology, using either STot or PTot as the totalisation operator. We sometimes call the theory obtained with STot standard [1] to distinguish it from the theory obtained with PTot which we call completed cohomology of DG sheaves. We add some elementary properties, which we need in Appendix A and which are routine generalisations of either the case of ungraded sheaves or DG sheaves which are bounded below. For a sheaf $\mathcal{L}$ on X let us denote by $C^0(-,\mathcal{L})$ the presheaf of serrations of $\mathcal{L}$, given by

$$C^0(U;\mathcal{L}) = \prod_{x\in U} \mathcal{L}_x.$$

This presheaf is a sheaf. For this particular sheaf let us adopt special conventions. Let us write its value on an open set U as $C^0(U;\mathcal{L})$, but when we consider it as a sheaf on X, we write it as $\mathcal{C}^0(X;\mathcal{L})$. This construction is functorial in the following sense. Every sheaf homomorphism $k : \mathcal{B} \to \mathcal{A}$ over a map $f : X \to Y$ induces R-module homomorphisms $k_x : \mathcal{B}_{f(x)} \to \mathcal{A}_x$ and thus also a homomorphism of sheaves over f given by

$$C^0(V;\mathcal{B}) = \prod_{y\in V} \mathcal{B}_y \longrightarrow \prod_{x\in f^{-1}V} \mathcal{A}_x = C^0(f^{-1}V;\mathcal{A}).$$

Definition 4.2.1. A sheaf $\mathcal{L}$ on a space X is called *flabby* if the restriction homomorphism $\mathcal{L}(X) \to \mathcal{L}(U)$ is onto for every open set U in X.

[1] In Appendix A we show that this cohomology is isomorphic to Čech cohomology of DG sheaves on compact spaces. It is not difficult to prove that this is true for DG sheaves on any metrisable space.

The sheaf $\mathcal{C}^0(X;\mathcal{L})$ is obviously flabby for every sheaf $\mathcal{L}$ and space X.

There is a canonical injection $j : \mathcal{L} \rightarrowtail \mathcal{C}^0(X;\mathcal{L})$ given by the inclusion $\mathcal{L}(U) \hookrightarrow C^0(U;\mathcal{L})$ for every open set U in X. This injection commutes with sheaf morphisms.

In the case of a DG sheaf $\mathcal{L}^\bullet$ the sequence $\mathcal{C}^0(X;\mathcal{L}^\bullet)$ inherits the structure of a DG sheaf, $j : \mathcal{L}^\bullet \hookrightarrow \mathcal{C}^0(X;\mathcal{L}^\bullet)$ is map of DG sheaves. Every sheaf $\mathcal{C}^0(X;\mathcal{L}^q)$ is flabby and the construction is functorial.

Let us define inductively the DG sheaves $\mathcal{C}^n(X;\mathcal{L}^\bullet)$. First we define

$$\mathcal{Z}^1(X;\mathcal{L}^n) = \mathcal{C}^0(X;\mathcal{L}^n)/\mathcal{L}^n.$$

The differential d in $\mathcal{L}^\bullet$ induces a differential map in the quotient sheaves which makes $\mathcal{Z}^1(X;\mathcal{L}^\bullet)$ a DG sheaf. Set

$$\begin{array}{ll} \mathcal{C}^1(X;\mathcal{L}^\bullet) = \mathcal{C}^0(X;\mathcal{Z}^1(X;\mathcal{L}^\bullet)) & \mathcal{Z}^2(X;\mathcal{L}^\bullet) = \mathcal{C}^1(X;\mathcal{L}^\bullet)/\mathcal{Z}^1(X;\mathcal{L}^\bullet) \\ \dots & \dots \\ \mathcal{C}^n(X;\mathcal{L}^\bullet) = \mathcal{C}^0(X;\mathcal{Z}^n(X;\mathcal{L}^\bullet)) & \mathcal{Z}^{n+1}(X;\mathcal{L}^\bullet) = \mathcal{C}^n(X;\mathcal{L}^\bullet)/\mathcal{Z}^n(X;\mathcal{L}^\bullet) \\ \dots & \dots \end{array}$$

All the sheaves $\mathcal{C}^n(X;\mathcal{L}^m)$ are flabby.

There are sheaf morphisms $\delta : \mathcal{C}^n(X;\mathcal{L}^\bullet) \to \mathcal{C}^{n+1}(X;\mathcal{L}^\bullet)$ defined as the composition

$$\mathcal{C}^n(X;\mathcal{L}^\bullet) \twoheadrightarrow \mathcal{C}^n(X;\mathcal{L}^\bullet)/\mathcal{Z}^n(X;\mathcal{L}^\bullet) \hookrightarrow \mathcal{C}^0(X;\mathcal{C}^n(X;\mathcal{L}^\bullet)/\mathcal{Z}^n(X;\mathcal{L}^\bullet))$$

δ commutes with d as both the quotient and the inclusion maps are maps of DG sheaves. It is easy to check that δ is a differential,i.e. $\delta\delta = 0$, and that the sequence

$$0 \to \mathcal{L}^\bullet \hookrightarrow \mathcal{C}^0(X;\mathcal{L}^\bullet) \xrightarrow{\delta} \mathcal{C}^1(X;\mathcal{L}^\bullet) \xrightarrow{\delta} \mathcal{C}^2(X;\mathcal{L}^\bullet) \xrightarrow{\delta} \cdots$$

is exact.

The DG sheaves $\mathcal{C}^*(-,-)$ are functorial with respect to sheaf homomorphisms over maps and therefore also with respect to homomorphisms of DG sheaves over maps.

There is a double complex $C^*(X;\mathcal{L}^\bullet)$ with the (p,q)-entry being

$$C^p(X;\mathcal{L}^q)$$

and with the differentials δ and d induced in the usual way by the ones defined above.

Proposition 4.2.2. *The functors $\mathcal{L}^\bullet \mapsto \mathcal{C}^n(X;\mathcal{L}^\bullet)$ and $\mathcal{L}^\bullet \mapsto C^n(X;\mathcal{L}^\bullet)$ on the category of DG sheaves on X are exact.*

Proof of 4.2.2: If

$$0 \to \mathcal{L}^\bullet \to \mathcal{M}^\bullet \to \mathcal{N}^\bullet \to 0$$

is an exact sequence of DG sheaves, then

$$0 \to \prod_{x \in U} \mathcal{L}^\bullet_x \to \prod_{x \in U} \mathcal{M}^\bullet_x \to \prod_{x \in U} \mathcal{N}^\bullet_x \to 0$$

is an exact sequence of cochain complexes, for every open set U in X. Therefore the exact sequence of DG sheaves

$$0 \to \mathcal{C}^0(X;\mathcal{L}^\bullet) \to \mathcal{C}^0(X;\mathcal{M}^\bullet) \to \mathcal{C}^0(X;\mathcal{N}^\bullet) \to 0$$

as well as the exact sequence of cochain complexes

$$0 \to C^0(X;\mathcal{L}^\bullet) \to C^0(X;\mathcal{M}^\bullet) \to C^0(X;\mathcal{N}^\bullet) \to 0$$

are both exact. Passing to the quotient one obtains an exact sequence

$$0 \to \mathcal{Z}^1(X;\mathcal{L}^\bullet) \to \mathcal{Z}^1(X;\mathcal{M}^\bullet) \to \mathcal{Z}^1(X;\mathcal{N}^\bullet) \to 0\,.$$

By induction we obtain exactness of $\mathcal{L}^\bullet \mapsto \mathcal{C}^n(X;\mathcal{L}^\bullet)$, $\mathcal{L}^\bullet \mapsto \mathcal{Z}^n(X;\mathcal{L}^\bullet)$ and therefore also $\mathcal{L}^\bullet \mapsto C^n(X;\mathcal{L}^\bullet) = C^0(X;\mathcal{Z}^n(X;\mathcal{L}^\bullet))$. As the construction of the differential δ is functorial, we obtain the result. $\boxed{4.2.2}$

Remark: A sheaf $\mathcal{L}$ can be regarded as a DG sheaf in the natural way ($\mathcal{L}^0 = \mathcal{L}$ and $\mathcal{L}^\bullet = 0$ for $\bullet \neq 0$). Therefore it follows from the proposition above that in particular the functor $\mathcal{L} \mapsto C^n(X;\mathcal{L})$ is exact.

Definition 4.2.3. We call a DG sheaf $\mathcal{L}^\bullet$ *acyclic* if its cohomology DG sheaf is trivial.

Corollary 4.2.4. *Let $\mathcal{L}^\bullet$ be an acyclic DG sheaf. Then $E_1(C^*(X;\mathcal{L}^\bullet)) = 0$ for every $*$ and $\bullet$.*

Proof of 4.2.4: Let $\mathcal{K}^n = \mathrm{Ker}(d : \mathcal{L}^n \to \mathcal{L}^{n+1})$ and $\mathcal{B}^n = \mathrm{Im}(d : \mathcal{L}^{n-1} \to \mathcal{L}^n)$. There are short exact sequences of sheaves

$$0 \to \mathcal{K}^n \to \mathcal{L}^n \to \mathcal{B}^{n+1} \to 0$$

$$0 \to \mathcal{B}^n \to \mathcal{K}^n \to \mathcal{H}^n \to 0\,.$$

As C^* is an exact functor on the category of sheaves on X

$$\mathrm{Im}(d : C^*\mathcal{L}^{n-1} \to C^*\mathcal{L}^n) = C^*\mathcal{B}^n = C^*\mathcal{K}^n = \mathrm{Ker}(d : C^*\mathcal{L}^n \to C^*\mathcal{L}^{n+1})\,,$$

the vertical differential in $C^*\mathcal{L}^\bullet$ is exact and so $E_1(C^*\mathcal{L}^\bullet) = 0$. $\boxed{4.2.4}$

Definition 4.2.5. Let $\mathcal{L}^\bullet$ be a DG sheaf on a space X. Then we define its *cohomology*, $H^*(X;\mathcal{L}^\bullet)$, to be

$$H^*(X;\mathcal{L}^\bullet) = \mathrm{H}(\mathrm{STot}(C^*(X;\mathcal{L}^\bullet)))$$

and its *completed cohomology*, $\hat{H}^*(X;\mathcal{L}^\bullet)$, to be

$$\hat{H}^*(X;\mathcal{L}^\bullet) = \mathrm{H}(\mathrm{PTot}(C^*(X;\mathcal{L}^\bullet)))\,.$$

Therefore $\hat{H}^*(X;\mathcal{L}^\bullet)$ is functorial with respect to DG sheaf homomorphisms over maps and there is a natural map

$$\kappa : H^*(X;\mathcal{L}^\bullet) \longrightarrow \hat{H}^*(X;\mathcal{L}^\bullet),$$

satisfying Axiom 3.4.1(a).

We shall explore the properties of the completed cohomology in the next sections of this chapter, here let us mention only two classical results about the cohomology of DG sheaves which we shall need later.

Definition 4.2.6. A sheaf $\mathcal{A}$ on a space X is called *δ-acyclic sheaf* if

$$H^i(X;\mathcal{A}) = \mathrm{H}^i(C^*(X;\mathcal{A})) = 0$$

for all $i > 0$.

In particular every flabby, fine or soft sheaf is a δ-acyclic sheaf.

Theorem 4.2.7. *If the DG sheaf $\mathcal{L}^\bullet$ on X is such that every component $\mathcal{L}^n$ is a δ-acyclic sheaf, then*

$$H^*(X;\mathcal{L}^\bullet) = \mathrm{H}(\mathcal{L}^\bullet(X))\,.$$

Proof of 4.2.7: The assumptions imply that the double complex $C^*(X;\mathcal{L}^\bullet)$ is exact for the horizontal differential δ. It follows by a standard double complex argument that every cohomology class of degree n in $\mathrm{H}(\mathrm{STot}(C^*(X;\mathcal{L}^\bullet)))$ can be represented by an element $\gamma \in C^0(X;\mathcal{L}^n)$ such that $\delta\gamma = 0$ and $d\gamma = 0$. The exact sequence

4.2.8. $$0 \longrightarrow \mathcal{L}^n(X) \longrightarrow C^0(X;\mathcal{L}^n) \longrightarrow C^1(X;\mathcal{L}^n)$$

implies that $\gamma \in \mathcal{L}^n(X)$ and that the homomorphism $H^n(X;\mathcal{L}^\bullet) \to \mathrm{H}(\mathcal{L}^n(X))$ is the inverse of the homomorphism induced by $\mathcal{L}^n(X) \to C^0(X;\mathcal{L}^n)$. $\boxed{4.2.7}$

Theorem 4.2.9. *Let*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathfrak{L}^{n+1} & \to & \mathfrak{D}^{0,n+1} & \to & \mathfrak{D}^{1,n+1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathfrak{L}^{n} & \to & \mathfrak{D}^{0,n} & \to & \mathfrak{D}^{1,n} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathfrak{L}^{n-1} & \to & \mathfrak{D}^{0,n-1} & \to & \mathfrak{D}^{1,n-1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

be a commutative diagram of sheaves such that all columns are DG sheaves and all rows are δ-acyclic resolutions. Then there is a canonical isomorphism

$$\mathrm{H}(\mathrm{STot}(D^{\bullet,\bullet})) = H^*(X;\mathfrak{L}^\bullet)$$

where $D^{\bullet,\bullet} = \mathfrak{D}^{\bullet,\bullet}(X)$ and the differentials are induced in the standard way.

Proof of 4.2.9: Let us denote the horizontal differentials by

$$\delta : \mathfrak{D}^{\bullet,\bullet} \to \mathfrak{D}^{\bullet+1,\bullet}.$$

The assumptions imply that $C^*(X;\mathfrak{D}^{\bullet,\bullet})$ is a triple complex. The differentials

$$\delta : C^*(X;\mathfrak{D}^{\bullet,\bullet}) \longrightarrow C^{*+1}(X;\mathfrak{D}^{\bullet,\bullet}), \quad \delta : C^*(X;\mathfrak{D}^{\bullet,\bullet}) \longrightarrow C^*(X;\mathfrak{D}^{\bullet+1,\bullet})$$

are both exact. There are injections

$$\epsilon : D^{\bullet,\bullet} \longrightarrow C^*(X;\mathfrak{D}^{\bullet,\bullet}), \quad \eta : C^*(X;\mathfrak{L}^\bullet) \longrightarrow C^*(X;\mathfrak{D}^{\bullet,\bullet})$$

which make the following sequences exact:

$$0 \to D^{\bullet,\bullet} \xrightarrow{\epsilon} C^0(X;\mathfrak{D}^{\bullet,\bullet}) \xrightarrow{\delta} C^1(X;\mathfrak{D}^{\bullet,\bullet})$$

$$0 \to C^*(X;\mathfrak{L}^\bullet) \xrightarrow{\eta} C^*(X;\mathfrak{D}^{0,\bullet}) \xrightarrow{\delta} C^*(X;\mathfrak{D}^{1,\bullet}).$$

To the triple complex $C^*(X;\mathfrak{D}^{\bullet,\bullet})$ we can naturally associate a single complex $\mathrm{STot}C^*(X;\mathfrak{D}^{\bullet,\bullet})$ with the homogeneous elements of degree n being

$$\bigoplus_{p+q+r=n} C^p(X;\mathfrak{D}^{q,r})$$

and the total differential on $C^p(X;\mathfrak{D}^{q,r})$ is $\delta + (-1)^p\delta + (-1)^{p+q}d$.

Because of exactness of δ, a standard argument shows that every cohomology class in $\mathrm{STot}C^*(X;\mathfrak{D}^{\bullet,\bullet})$ can be represented by a cocycle in the image of ϵ. Thus in cohomology ϵ induces

$$\mathrm{H}(\mathrm{STot}(C^*(X;\mathfrak{D}^{\bullet,\bullet}))) \cong \mathrm{H}(\mathrm{STot}(D^{\bullet,\bullet})).$$

Similarly η induces

$$H^*(X;\mathfrak{L}^\bullet) \cong \mathrm{H}(\mathrm{STot}(D^{\bullet,\bullet})).$$

$\boxed{4.2.9}$

4.3. Dimension

In this section we prove Axiom 3.4.1(b) of the completed cohomology of DG sheaves, i.e. that it agrees with the standard cohomology on all spaces with finite cohomological dimension. Here we need the lemmas proved in Section 4.1.

Theorem 4.3.1. *If the space X has finite cohomological dimension, the natural map*

$$\kappa : H^{*}(X;\mathcal{L}^{\bullet}) \longrightarrow \hat{H}^{*}(X;\mathcal{L}^{\bullet})$$

is an isomorphism.

Proof of 4.3.1: If $\mathrm{cd}X \leq n$, then the exact sequence of sheaves

$$0 \to \mathcal{L} \longrightarrow \mathcal{C}^{0}\mathcal{L} \longrightarrow \mathcal{C}^{1}\mathcal{L} \to \cdots \to \mathcal{C}^{n-1}\mathcal{L} \longrightarrow \mathcal{Z}^{n}\mathcal{L} \to 0$$

is a δ-acyclic resolution of $\mathcal{L}$, for any sheaf $\mathcal{L}$ on X. Let us denote this resolution by $\mathcal{F}^{*}\mathcal{L}$ and $\mathcal{F}^{*}\mathcal{L}(X)$ by $F^{*}\mathcal{L}$. Then F^{*} is an exact functor on the category of sheaves on X [Godement 58, p. 195]. For the DG sheaf $\mathcal{L}^{\bullet}$ there is a canonical injection $F^{*}\mathcal{L}^{\bullet} \hookrightarrow C^{*}\mathcal{L}^{\bullet}$ and therefore a short exact sequence of double complexes

$$0 \to F^{*}\mathcal{L}^{\bullet} \longrightarrow C^{*}\mathcal{L}^{\bullet} \longrightarrow Q^{*,\bullet} \to 0 \,.$$

This short exact sequence induces homomorphisms of the E_1 terms of the double complexes

$$E_1(F)^{p,q} \longrightarrow E_1(C)^{p,q} \longrightarrow E_1(Q)^{p,q}$$

such that for every p we have a long exact sequence

4.3.2. $$\cdots \longrightarrow E_1(F)^{p,q} \longrightarrow E_1(C)^{p,q} \longrightarrow E_1(Q)^{p,q} \longrightarrow E_1(F)^{p,q+1} \longrightarrow \cdots \,.$$

Let $\mathcal{K}^{\bullet} \leq \mathcal{L}^{\bullet}$ denote the DG subsheaf of cocycles and $\mathcal{B}^{\bullet} \leq \mathcal{L}^{\bullet}$ the DG subsheaf of coboundaries. Let $\mathcal{H}^{\bullet}$ denote the cohomology DG sheaf of $\mathcal{L}^{\bullet}$. Because of exactness of functors F^{*} and C^{*} there are commutative diagrams with exact rows

$$\begin{tikzcd}
 & & & F^{\cdot}\mathcal{L}^{\bullet+1} \arrow[dd] & & \\
0 \arrow[r] & F^{\cdot}\mathcal{K}^{\bullet} \arrow[r] \arrow[dd] & F^{\cdot}\mathcal{L}^{\bullet} \arrow[rr] \arrow[ur] \arrow[dd] & & F^{\cdot}\mathcal{B}^{\bullet+1} \arrow[r] \arrow[ul] \arrow[dd] & 0 \\
 & & & C^{\cdot}\mathcal{L}^{\bullet+1} & & \\
0 \arrow[r] & C^{\cdot}\mathcal{K}^{\bullet} \arrow[r] & C^{\cdot}\mathcal{L}^{\bullet} \arrow[rr] \arrow[ur] & & C^{\cdot}\mathcal{B}^{\bullet+1} \arrow[r] \arrow[ul] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & F^{\cdot}\mathcal{K}^{\bullet}/F^{\cdot}\mathcal{B}^{\bullet} \arrow[dd] & & \\
0 \arrow[r] & F^{\cdot}\mathcal{B}^{\bullet} \arrow[r] \arrow[dd] & F^{\cdot}\mathcal{K}^{\bullet} \arrow[rr] \arrow[ur] \arrow[dd] & & F^{\cdot}\mathcal{H}^{\bullet} \arrow[r] \arrow[ul, "\cong"'] \arrow[dd] & 0 \\
 & & & C^{\cdot}\mathcal{K}^{\bullet}/C^{\cdot}\mathcal{B}^{\bullet} & & \\
0 \arrow[r] & C^{\cdot}\mathcal{B}^{\bullet} \arrow[r] & C^{\cdot}\mathcal{K}^{\bullet} \arrow[rr] \arrow[ur] & & C^{\cdot}\mathcal{H}^{\bullet} \arrow[r] \arrow[ul, "\cong"'] & 0\,.
\end{tikzcd}$$

Therefore we can identify $E_1(F)^{p,q} = F^p\mathcal{H}^q$, $E_1(C)^{p,q} = C^p\mathcal{H}^q$ and the homomorphism $E_1(F)^{p,q} \rightarrowtail E_1(C)^{p,q}$ is the injection $F^p\mathcal{H}^q \hookrightarrow C^p\mathcal{H}^q$. Therefore the coboundary homomorphisms in the long exact sequences 4.3.2 are all trivial. Thus we have a short exact sequence

$$0 \to E_1(F)^{\cdot,\bullet} \longrightarrow E_1(C)^{\cdot,\bullet} \longrightarrow E_1(Q)^{\cdot,\bullet} \to 0$$

and therefore for every $q \in \mathbb{Z}$ we have a long exact sequence

$$\cdots \longrightarrow E_2(F)^{p,q} \longrightarrow E_2(C)^{p,q} \longrightarrow E_2(Q)^{p,q} \longrightarrow E_2(F)^{p+1,q} \longrightarrow \cdots.$$

Because of $E_2(F)^{p,q} = H^p(X;\mathcal{H}^q) = E_2(C)^{p,q}$ we are in the position to apply Lemma 4.1.3 and obtain the following isomorphism

$$\check{H}^{\cdot}(X;\mathcal{L}^{\bullet}) \cong \mathrm{H}(\mathrm{PTot}(F^{\cdot}\mathcal{L}^{\bullet}))\,.$$

Obviously $\mathrm{PTot}(F^{\cdot}\mathcal{L}^{\bullet}) = \mathrm{STot}(F^{\cdot}\mathcal{L}^{\bullet})$ and since Theorem 4.2.9 gives a canonical isomorphism

$$H^{\cdot}(X;\mathcal{L}^{\bullet}) \cong \mathrm{H}(\mathrm{STot}(F^{\cdot}\mathcal{L}^{\bullet}))$$

we obtain the result. $\boxed{4.3.1}$

4.4. Exact sequence of a pair

Here we prove Axiom 3.4.1(c). We adapt the usual exact sequence of a pair to our case in an essentially routine manner. For a DG sheaf $\mathcal{L}^{\bullet}$ on X and an open or closed subspace A of X we denote by $\mathcal{L}^{\bullet}_A$ the unique sheaf on X which, when restricted to A, is $\mathcal{L}^{\bullet}|_A$ and is 0 on $X - A$. For a closed subspace A in X we sometimes write $\mathcal{L}^{\bullet}_{(X,A)}$ instead of $\mathcal{L}^{\bullet}_{X-A}$.

Theorem 4.4.1. *If $\mathcal{L}^{\bullet}$ is a differential graded sheaf on a space X and A is a closed subspace of X, then there is a long exact sequence*

$$\cdots \longrightarrow \hat{H}^{\cdot}(X;\mathcal{L}^{\bullet}_{(X,A)}) \longrightarrow \hat{H}^{\cdot}(X;\mathcal{L}^{\bullet}) \longrightarrow \hat{H}^{\cdot}(X;\mathcal{L}^{\bullet}_A)$$
$$\longrightarrow \hat{H}^{\cdot+1}(X;\mathcal{L}^{\bullet}_{(X,A)}) \longrightarrow \cdots$$

and this construction is functorial with respect to homomorphisms of DG sheaves over maps.

Proof of 4.4.1: There is a short exact sequence of DG sheaves

$$0 \to \mathcal{L}^{\bullet}_{(X,A)} \longrightarrow \mathcal{L}^{\bullet} \longrightarrow \mathcal{L}^{\bullet}_A \to 0$$

inducing a short exact sequence of double complexes

$$0 \to C^{\cdot}\mathcal{L}^{\bullet}_{(X,A)} \longrightarrow C^{\cdot}\mathcal{L}^{\bullet} \longrightarrow C^{\cdot}\mathcal{L}^{\bullet}_A \to 0$$

and this in turn induces the long exact sequence of the cohomology of the total complexes.

If $f : (X,A) \to (Y,B)$ is a map of pairs, $\bar{f} : X \to Y$ the underlying map of spaces and $k : \mathcal{K}^{\bullet} \to \mathcal{L}^{\bullet}$ is a DG sheaf homomorphism over $\bar{f}$, it induces a morphism of short exact sequences of DG sheaves

$$\begin{array}{ccccccc}
0 \to & \mathcal{K}^{\bullet}_{(Y,B)} & \longrightarrow & \mathcal{K}^{\bullet} & \longrightarrow & \mathcal{K}^{\bullet}_B & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{L}^{\bullet}_{(X,A)} & \longrightarrow & \mathcal{L}^{\bullet} & \longrightarrow & \mathcal{L}^{\bullet}_A & \to 0
\end{array}$$

and this in turn induces a morphism of short exact sequences of double complexes and the resulting morphism of the long exact sequences. $\boxed{4.4.1}$

Lemma 4.4.2. *For a closed subspace A of X and a DG sheaf $\mathcal{L}^{\bullet}$ on X there is a canonical isomorphism $H^{\cdot}(X;\mathcal{L}^{\bullet}_A) \cong H^{\cdot}(A;\mathcal{L}^{\bullet}_A)$.*

Proof of 4.4.2: The stalks of $\mathcal{C}^*(X;\mathcal{L}^\bullet_A)$ are as follows

$$\mathcal{C}^*(X;\mathcal{L}^\bullet_A)_x = \begin{cases} \mathcal{C}^*(A;\mathcal{L}^\bullet_A)_x & \text{if } x \in A \\ 0 & \text{if } x \in X - A \end{cases}$$

and therefore we obtain a canonical isomorphism $C^*(X;\mathcal{L}^\bullet_A) = C^*(A;\mathcal{L}^\bullet_A)$ and the result. $\boxed{4.4.2}$

4.5. Quasi-isomorphisms

Here we prove Axiom 3.4.1(d) of the completed cohomology. This property is true for the cohomology of positively graded differential sheaves but not necessarily for the standard cohomology of DG sheaves which are not bounded below if the underlying space is infinite dimensional.

Proposition 4.5.1. *If the DG sheaf $\mathcal{L}^\bullet$ is acyclic, then $\hat{H}^*(X;\mathcal{L}^\bullet) = 0$.*

Proof of 4.5.1: By Corollary 4.2.4 we obtain $E_1(C^*\mathcal{L}^\bullet) = 0$. Lemma 4.1.2 implies the result. $\boxed{4.5.1}$

Theorem 4.5.2. *If $f : \mathcal{L}^\bullet \to \mathcal{M}^\bullet$ is a homomorphism of differential graded sheaves on a space X, which induces an isomorphism of the respective cohomology DG sheaves $f : \mathcal{H}^\bullet(\mathcal{L}) \to \mathcal{H}^\bullet(\mathcal{M})$ then f induces an isomorphism*

$$f : \hat{H}^*(X;\mathcal{L}^\bullet) \xrightarrow{\cong} \hat{H}^*(X;\mathcal{M}^\bullet).$$

Proof of 4.5.2: Construct a DG sheaf $\mathcal{K}^\bullet_f$ with $\mathcal{K}^n_f = \mathcal{L}^n \oplus \mathcal{M}^{n-1}$ and the differential $(-d_{\mathcal{L}}, d_{\mathcal{M}} + f)$. Let $S\mathcal{M}^\bullet$ be defined by $S\mathcal{M}^n = \mathcal{M}^{n-1}$. There is a short exact sequence of DG sheaves

$$0 \to S\mathcal{M}^\bullet \longrightarrow \mathcal{K}^\bullet_f \longrightarrow \mathcal{L}^\bullet \to 0\ .$$

The assumptions imply that $\mathcal{K}^\bullet_f$ is acyclic, i.e. its cohomology DG sheaf is trivial: Let (l, m) be a cocycle in the stalk of $\mathcal{K}^n_f$ over a point $x \in X$ so that

$$d(l) = 0,\ d(m) = -f(l)\ .$$

Then the condition that f induce an isomorphism of the derived DG sheaves implies that there exists an element a in the stalk of $\mathcal{L}^{n-1}$ over x such that $d(a) = l$. As $f(a) + m$ is a cocycle in the stalk of $\mathcal{M}^{n-1}$ over x, there exists a cocycle b in the stalk of $\mathcal{L}^{n-1}$ over x such that $f(b) = f(a) + m$. Therefore

$$d(-a + b, 0) = (l, m)$$

and the cohomology of the stalk of $\mathcal{K}^\bullet_f$ over x is zero.

As $C^{\cdot}$ and PTot are exact functors we have an exact sequence

$$0 \to \mathrm{PTot}(C^{\cdot}\mathrm{S}\mathcal{M}^{\bullet}) \longrightarrow \mathrm{PTot}(C^{\cdot}\mathcal{K}_f^{\bullet}) \longrightarrow \mathrm{PTot}(C^{\cdot}\mathcal{L}^{\bullet}) \to 0$$

of cochain complexes and this in turn induces a long exact sequence in cohomology. Since $\hat{H}^{\cdot}(X;\mathcal{K}_f^{\bullet})$ is trivial it follows

$$\hat{H}^{\cdot}(X;\mathcal{L}^{\bullet}) \cong \hat{H}^{\cdot+1}(X;\mathrm{S}\mathcal{M}^{\bullet}) = \hat{H}^{\cdot}(X;\mathcal{M}^{\bullet}) .$$

$\boxed{4.5.2}$

4.6. Homotopy invariance

In this section we prove Axiom 3.4.1(e) of the completed cohomology of DG sheaves. The homotopy invariance of the standard cohomology of DG sheaves is essentially a straightforward generalisation of the nongraded case (Vietoris-Begle theorem). For the completed cohomology, however, the proof is not so obvious. Here we shall use the facts recollected in Section 4.1.

Theorem 4.6.1. *Let $\mathcal{L}^{\bullet}$ be a DG sheaf on a space X and $\pi^{\sharp}\mathcal{L}^{\bullet} = \mathcal{L}^{\bullet} \times I$ the pull-back DG sheaf on the product $X \times I$. Then the projection $\pi : X \times I \to X$ induces a DG sheaf homomorphism $\pi^{\bullet} : \mathcal{L}^{\bullet} \to \mathcal{L}^{\bullet} \times I$ over π and this homomorphism induces an isomorphism*

$$\hat{H}^{\cdot}(X;\mathcal{L}^{\bullet}) \longrightarrow \hat{H}^{\cdot}(X \times I;\mathcal{L}^{\bullet} \times I).$$

Proof of 4.6.1: First we show that π induces an injective homomorphism of double complexes

$$\pi^{\cdot,\bullet} : C^{\cdot}(X;\mathcal{L}^{\bullet}) \to C^{\cdot}(X \times I;\mathcal{L}^{\bullet} \times I) .$$

Then we shall show that it induces an injective homomorphism of the E_1 terms and an isomorphism of the E_2 terms in order to apply Lemma 4.1.3.

We can canonically identify the double complexes

$$C^{\cdot}(X;\mathcal{L}^{\bullet}) = \pi^{\sharp}\mathcal{C}^{\cdot}(X;\mathcal{L}^{\bullet})(X \times I) .$$

Let

$$i_t : X \to X \times I, \quad x \mapsto (x,t) .$$

Then the pull-back homomorphism $\pi^{\sharp,\bullet} : \mathcal{L}^{\bullet} \to \mathcal{L}^{\bullet} \times I$ over π has left inverses $i_t^{\sharp,\bullet}$, for every $t \in I$. As $\mathcal{C}^{\cdot}(-,-)$ is a functor on sheaves and homomorphisms over maps, we obtain

$$i_t^{\cdot,\bullet} \circ \pi^{\cdot,\bullet} = \mathrm{Id}_{\mathcal{C}^{\cdot}(X;\mathcal{L}^{\bullet})}$$

for every $t \in I$, where $i_t^{*,\bullet}$ denotes $\mathcal{C}^*(i_t, i_t^{\sharp,\bullet})$ and $\pi^{*,\bullet}$ denotes $\mathcal{C}^*(\pi, \pi^{\sharp,\bullet})$. This implies that the homomorphisms

$$\pi_{(x,t)}^{*,\bullet} : \mathcal{C}^*(X; \mathcal{L}^\bullet)_x \to \mathcal{C}^*(X \times I; \mathcal{L}^\bullet \times I)_{(x,t)}$$

are injective for every $x \in X$ and $t \in I$. Every homomorphism of sheaves factorises through the pull-back and therefore $\pi^{*,\bullet}$ factorises as

4.6.2. $$\mathcal{C}^*(X; \mathcal{L}^\bullet) \to \pi^\sharp\mathcal{C}^*(X; \mathcal{L}^\bullet) \to \mathcal{C}^*(X \times I; \mathcal{L}^\bullet \times I)\,.$$

For every $x \in X$ and $t \in I$ the homomorphism $\pi_{(x,t)}^{*,\bullet}$ is injective and

$$\mathcal{C}^*(X; \mathcal{L}^\bullet)_x \to \pi^\sharp\mathcal{C}^*(X; \mathcal{L}^\bullet)_{(x,t)}$$

is an isomorphism. Therefore also

$$\pi^\sharp\mathcal{C}^*(X; \mathcal{L}^\bullet) \to \mathcal{C}^*(X \times I; \mathcal{L}^\bullet \times I)$$

is injective. The factorisation (4.6.2) gives the following factorisation of the double complex homomorphism

$$\pi^{*,\bullet} : C^*(X; \mathcal{L}^\bullet) \xrightarrow{=} \pi^\sharp\mathcal{C}^*(X; \mathcal{L}^\bullet)(X \times I) \rightarrowtail C^*(X \times I; \mathcal{L}^\bullet \times I)).$$

Let us show that also the induced homomorphism of the E_1 terms of the first spectral sequences of these double complexes is injective. Let $\mathcal{K}^\bullet \leq \mathcal{L}^\bullet$ be the DG subsheaf of cocycles and $\mathcal{B}^\bullet \leq \mathcal{L}^\bullet$ be the DG subsheaf of coboundaries. The commutative diagrams of homomorphisms

$$\begin{array}{ccccccccc} 0 \to & \mathcal{K}^\bullet & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{B}^{\bullet+1} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{K}^\bullet \times I & \longrightarrow & \mathcal{L}^\bullet \times I & \longrightarrow & \mathcal{B}^{\bullet+1} \times I & \to 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 \to & \mathcal{B}^\bullet & \longrightarrow & \mathcal{K}^\bullet & \longrightarrow & \mathcal{H}^\bullet & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{B}^\bullet \times I & \longrightarrow & \mathcal{K}^\bullet \times I & \longrightarrow & \mathcal{H}^\bullet \times I & \to 0 \end{array}$$

with exact rows induce the following commutative diagrams of double complexes.

$$\begin{tikzcd}
 & & & C^*\mathcal{L}^{\bullet+1} \arrow[dd] & & \\
0 \arrow[r] & C^*\mathcal{K}^\bullet \arrow[r] \arrow[dd] & C^*\mathcal{L}^\bullet \arrow[ur] \arrow[rr] \arrow[dd] & & C^*\mathcal{B}^{\bullet+1} \arrow[r] \arrow[ul] \arrow[dd] & 0 \\
 & & & C^*\mathcal{L}^{\bullet+1} \times I & & \\
0 \arrow[r] & C^*\mathcal{K}^\bullet \times I \arrow[r] & C^*\mathcal{L}^\bullet \times I \arrow[ur] \arrow[rr] & & C^*\mathcal{B}^{\bullet+1} \times I \arrow[r] \arrow[ul] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & C^*\mathcal{K}^\bullet / C^*\mathcal{B}^\bullet \arrow[dd] & & \\
0 \arrow[r] & C^*\mathcal{B}^\bullet \arrow[r] \arrow[dd] & C^*\mathcal{K}^\bullet \arrow[ur] \arrow[rr] \arrow[dd] & & C^*\mathcal{H}^\bullet \arrow[r] \arrow[ul, "\cong"'] \arrow[dd] & 0 \\
 & & & C^*\mathcal{K}^\bullet \times I / C^*\mathcal{B}^\bullet \times I & & \\
0 \arrow[r] & C^*\mathcal{B}^\bullet \times I \arrow[r] & C^*\mathcal{K}^\bullet \times I \arrow[ur] \arrow[rr] & & C^*\mathcal{H}^\bullet \times I \arrow[r] \arrow[ul, "\cong"'] & 0
\end{tikzcd}$$

where $C^*\mathcal{L}^\bullet = C^*(X;\mathcal{L}^\bullet)$, $C^*\mathcal{L}^\bullet \times I = C^*(X \times I;\mathcal{L}^\bullet \times I)$ and similarly for the other DG sheaves. Therefore the induced homomorphism of the E_1 terms of the double complexes can be identified with $C^*(X;\mathcal{H}^\bullet) \to C^*(X \times I;\mathcal{H}^\bullet \times I)$ which is injective by the first part of the proof. The induced homomorphism of the E_2 terms is

$$\pi^{*,\bullet} : H_\delta^*(X;\mathcal{H}^\bullet) \longrightarrow H_\delta^*(X \times I;\mathcal{H}^\bullet \times I)$$

which is an isomorphism by the usual homotopy invariance [Bredon 65, p. 56]. Therefore we can apply Lemma 4.1.3 and obtain the result. $\boxed{4.6.1}$

4.7. Excision

In this section we prove Axiom 3.4.1(f) of the completed cohomology. The excision property in its standard form, i.e.

$$\hat{H}^*(X;\mathcal{L}^\bullet_{(X,A)}) \cong \hat{H}^*(X - U;\mathcal{L}^\bullet_{(X-U,A-U)})$$

for an open subspace U of X such that $U \subseteq A$ is trivially true as there is a canonical isomorphism between the corresponding cochain complexes. However, the completed cohomology of DG sheaves has a stronger form of the excision property:

Theorem 4.7.1. *Let $\mathcal{L}^\bullet$ be a DG sheaf on a space X and let U be an open subspace of X such that the cohomology DG sheaf $\mathcal{H}^\bullet$ of $\mathcal{L}^\bullet$ is trivial on U. Then the restriction of $\mathcal{L}^\bullet$ to $X - U$ induces an isomorphism*

$$\breve{H}^*(X; \mathcal{L}^\bullet) \cong \breve{H}^*(X - U; \mathcal{L}^\bullet_{X-U}).$$

Proof of 4.7.1: In the cohomology long exact sequence of the pair $(X, X - U)$ the term $\breve{H}^*(X; \mathcal{L}^\bullet_U)$ is trivial by Corollary 4.2.4 and Lemma 4.1.2. By Lemma 4.4.2 there is a canonical isomorphism $\breve{H}^*(X; \mathcal{L}^\bullet_{X-U}) \cong \breve{H}^*(X - U; \mathcal{L}^\bullet_{X-U})$. $\boxed{4.7.1}$

4.8. Milnor's additivity

Here we prove Axiom 3.4.1(g) of the completed cohomology.

Lemma 4.8.1. *If $X = \coprod_{\lambda \in \Lambda} X_\lambda$, then*

$$\breve{H}^*(X; \mathcal{L}^\bullet) \cong \prod_{\lambda \in \Lambda} \breve{H}^*(X_\lambda; \mathcal{L}^\bullet_{X_\lambda})$$

Proof of 4.8.1: There is a canonical isomorphism $\mathcal{L}^\bullet \cong \prod_\lambda \mathcal{L}^\bullet_{X_\lambda}$ inducing a canonical isomorphism $C^*(X; \mathcal{L}^\bullet) = \prod_\lambda C^*(X_\lambda; \mathcal{L}^\bullet_{X_\lambda})$ of double complexes, which gives the result. $\boxed{4.8.1}$

Chapter 5

Cellular decompositions

In this chapter we show that Jones-Petrack cohomology agrees with localised equivariant cohomology on all (not necessarily compact) finite dimensional $\mathbf{T}$-complexes. Since Jones-Petrack cohomology satisfies Milnor's axiom, we can relate $h_{\mathbf{T}}^*(X, A)$, for a pair of $\mathbf{T}$-complexes (X, A), with $\lim_{\leftarrow} u^{-1} H_{\mathbf{T}}^*(X^p, A)$. Then we show that there exist cellular versions of equivariant cohomology and localised equivariant cohomology.

5.1. The SJB exact sequence

This is an introductory section in which we describe a long exact sequence of $\mathbf{T}$-equivariant cohomology.

For every $\mathbf{T}$-pair (X, A) there exists an injection

$$\mathrm{S} : S^*(X, A)[u] \longrightarrow S^*(X, A)[u]$$

of degree 2, given by

$$\sum_{i \geq 0} \alpha_{p-2i} u^i \mapsto \sum_{i \geq 0} \alpha_{p-2i} u^{i+1}.$$

The quotient map is

$$\mathrm{J} : S^*(X, A)[u] \longrightarrow S^*(X, A)$$

of degree 0, given by

$$\sum_{i \geq 0} \alpha_{p-2i} u^i \mapsto \alpha_0 \, .$$

Therefore we have a short exact sequence of cochain complexes

$$0 \to \mathrm{Tot} S^*(X, A)[u] \xrightarrow{\mathrm{S}} \mathrm{Tot} S^*(X, A)[u] \xrightarrow{\mathrm{J}} S^*(X, A) \to 0$$

and let

$$\mathrm{B} : H^*(X, A) \longrightarrow H_{\mathbf{T}}^*(X, A)$$

of degree -1 be the induced coboundary map[1].

The operator B can be described as follows. Let $\alpha \in S^p(X, A)$ be a cocycle. Let us denote the differential map of singular cochains by ∂. Then α can be regarded as an element in $S^*(X, A)[u]$ and

$$\mathrm{B}([\alpha]) = [u^{-1}(\partial + u\mathrm{I})(\alpha)] = [\mathrm{I}\alpha]$$

which is an element of $H_{\mathbb{T}}^{p-1}(X, A)$.

The cochain operators S and J are natural and induce homomorphisms of long exact sequences of pairs. Also the operator B is natural in the sense that it commutes with cohomology homomorphisms induced by maps of pairs.

Proposition 5.1.1. *Let (X, A) be a pair of $\mathbb{T}$-spaces. Let δ be the coboundary homomorphism in the long exact sequence of $H^*(X, A)$ and $\delta_{\mathbb{T}}$ be the coboundary homomorphism in the long exact sequence of $H_{\mathbb{T}}^*(X, A)$. Then*

$$\mathrm{B}\delta = -\delta_{\mathbb{T}}\mathrm{B} \ .$$

Proof of 5.1.1: Let $\alpha \in S^p(A)$ be such that $\partial\alpha = 0$ and let $\gamma \in S^p(X)$ be such that $\gamma\,|_A = \alpha$. Then

$$\mathrm{B}\delta[\alpha] = [u^{-1}(\partial + u\mathrm{I})(\partial\gamma)] = [\mathrm{I}\partial\gamma]$$

and

$$\delta_{\mathbb{T}}\mathrm{B}[\alpha] = [(\partial + u\mathrm{I})(\mathrm{I}\gamma)] = [\partial\mathrm{I}\gamma].$$

The result follows since I and ∂ anticommute. $\boxed{5.1.1}$

Lemma 5.1.2. *Let (X, A) be a pair of $\mathbb{T}$-complexes. Then the diagram*

$$\begin{array}{ccccccc}
H^p(X) & \to & H^p(A) & & & & \\
\mathrm{J}\uparrow & & \uparrow & & & & \\
H_{\mathbb{T}}^p(X) & \xrightarrow{i_A^*} & H_{\mathbb{T}}^p(A) & \to & H_{\mathbb{T}}^{p+1}(X, A) & & \\
 & & \mathrm{S}\uparrow & & \uparrow & & \\
 & & H_{\mathbb{T}}^{p-2}(A) & \xrightarrow{\delta} & H_{\mathbb{T}}^{p-1}(X, A) & & \\
 & & \mathrm{B}\uparrow & & \uparrow & & \\
 & & H^{p-1}(A) & \to & H^p(X, A) & \xrightarrow{i_X^*} & H^p(X) \\
 & & & & \mathrm{J}\uparrow & & \\
 & & & & H_{\mathbb{T}}^p(X, A) & &
\end{array}$$

gives a homomorphism

$$\mathrm{Ker}(H_{\mathbb{T}}^p(X) \to H^p(A)) \longrightarrow H^p(X)/\mathrm{Im}(H_{\mathbb{T}}^p(X, A) \to H^p(X))$$

[1]Formally we can think of $\mathrm{Tot}S^*(X, A)[u]$ to have the degree raised by 2 in order for S to be of degree 0.

obtained by diagram chasing as $i_X^* B^{-1}\delta S^{-1} i_A^*$ which factorises as

$$\begin{array}{ccc} \mathrm{Ker}(H_{\mathbb{T}}^p(X) \to H^p(A)) & \longrightarrow & H^p(X)/\mathrm{Im}(H_{\mathbb{T}}^p(X,A) \to H^p(X)) \\ \downarrow & & \uparrow \\ H_{\mathbb{T}}^p(X) & \xrightarrow{J} & H^p(X). \end{array}$$

Proof of 5.1.2: The diagram shows that the homomorphism is well-defined. Let us show that it is determined by J.

Let

$$\alpha(u) = \sum_{j \in \mathbb{N}} \alpha_{p-2j} u^j$$

be a cocycle of degree p in $S^*(X)[u]$ such that $[\alpha(u)] \in \mathrm{Ker}(H_{\mathbb{T}}^p(X) \to H^p(A))$. Therefore there exists a singular cochain $\beta \in S^{p-1}(A)$ such that

$$\partial\beta = i^*\alpha_p$$

for the differential map $\partial : S^{p-1}(A) \to S^p(A)$ and $i^* : S^p(X) \to S^p(A)$. For

$$\alpha'(u) = (i^*\alpha_{p-2} - \mathrm{I}\beta) + ui^*\alpha_{p-4} + \ldots$$

we have

$$u\alpha'(u) + (\partial + u\mathrm{I})\beta = i^*(\alpha(u)).$$

The element $\delta[\alpha'(u)] \in H_{\mathbb{T}}^{p-1}(X, A)$ is given by the cocycle

$$\alpha''(u) = (\partial + u\mathrm{I})((\alpha_{p-2} - \mathrm{I}\bar{\beta}) + u\alpha_{p-4} + \ldots)$$

where $\bar{\beta} \in S^{p-1}(X)$ is such that $i^*(\bar{\beta}) = \beta$. Because $\alpha(u)$ was chosen to be a cocycle, we obtain

$$\alpha''(u) = \partial\alpha_{p-2} - \partial\mathrm{I}\bar{\beta} = \mathrm{I}\alpha_p - \partial\mathrm{I}\bar{\beta} = \mathrm{I}(\alpha_p + \partial\bar{\beta}).$$

Therefore $[\alpha_p + \partial\bar{\beta}] = [\alpha_p] \in H^p(X, A)$ is mapped by B to $\alpha''(u)$. The homomorphism is then given by

$$\alpha(u) \mapsto [\alpha_p] \in H^p(X)$$

which agrees with the homomorphism determined by J. $\boxed{5.1.2}$

5.2. Equivariant complexes

First we define G-complexes following [Willson 75]. Then we show that on all (not necessarily compact) finite dimensional T-complexes Jones-Petrack cohomology agrees with localised equivariant cohomology.

Definition 5.2.1. Let D^n denote the standard n-disk with boundary $\dot{D}^n$. Let H be a closed subgroup of a topological group G. The space

$$G/H \times D^n$$

is called a *G-cell* of type H and dimension n, or sometimes an n-cell of type H; it is a G-space with G-action $g_1 \cdot (g_2H, s) = (g_1g_2H, s)$ for $g_1, g_2 \in G$ and $s \in D^n$.

Note that the dimension of a G-cell need not equal its dimension as a topological space.

If X is a G-space and $f : G/H \times \dot{D}^n \to X$ a G-map, we obtain a new G-space Y by setting

$$Y = G/H \times D^n \cup_f X$$

i.e., by identifying points in $G/H \times \dot{D}^n \subset G/H \times D^n$ with their image in X. The G-space Y is said to be obtained from X by adjoining an n-cell.

Definition 5.2.2. Let Y be a G-space. A *relative G-complex* (X, Y) is a G-space X obtained inductively as follows: Let $X^{-1} = Y$. Define X^i to be the result of adjoining arbitrarily many G-cells of arbitrary type but dimension i to X^{i-1}; we give X^i the weak topology and the natural G-action. Let $X = \bigcup_i X^i$, with the weak topology. We call X^i the *i-skeleton* of the pair (X, Y). A *G-complex* X is a relative G-complex $(X, \emptyset)$. A *G-subcomplex* A of a G-complex X is a G-complex A such that (X, A) is a relative G-complex.

Let (X, A) be a relative $\mathbb{T}$-complex such that A is also a $\mathbb{T}$-complex. We call such a relative complex a pair of $\mathbb{T}$-complexes.

We shall need the following information about the cohomology of an n-cell e^n of type $\mathbb{Z}/k$.

$$H^*_{\mathbb{T}}(e^n, \partial e^n) = \begin{cases} \mathbb{Z} & * = n \\ (\mathbb{Z}/k)u^m & * = n + 2m \\ 0 & \text{else} \end{cases}$$

$$H^*(e^n, \partial e^n) = \begin{cases} \mathbb{Z} & \text{for } * = n, n+1 \\ 0 & \text{else.} \end{cases}$$

For a fixed point n-cell we have

$$H^*_{\mathbb{T}}(e^n, \partial e^n) = \begin{cases} \mathbb{Z}u^m & * = n + 2m \\ 0 & \text{else} \end{cases}$$

$$H^*(e^n, \partial e^n) = \begin{cases} \mathbb{Z} & \text{for } * = n \\ 0 & \text{else.} \end{cases}$$

From the SJB exact sequence we see that

$$JB : H^{n+1}(e^n, \partial e^n) \longrightarrow H^n(e^n, \partial e^n)$$

is multiplication by k if the n-cell e^n is of type $\mathbb{Z}/k$.

Proposition 5.2.3. *For every finite dimensional relative $\mathbb{T}$-complex (X^n, A) the natural map*

$$\epsilon : u^{-1}H^*_{\mathbb{T}}(X^n, A) \longrightarrow h^*_{\mathbb{T}}(X^n, A)$$

is an isomorphism.

Proof of 5.2.3:

Notice that even though localised equivariant cohomology does not satisfy Milnor's additivity axiom the inclusions of a family of n-cells in their disjoint union induce an isomorphism

$$u^{-1}H^*_{\mathbb{T}}(\coprod_\alpha (e^n_\alpha, \partial e^n_\alpha)) \longrightarrow \prod_\alpha u^{-1}H^*_{\mathbb{T}}(e^n_\alpha, \partial e^n_\alpha)$$

for every $n \geq 0$. Therefore we can identify $u^{-1}H^*_{\mathbb{T}}(X^n, X^{n-1})$ with $\prod_\alpha u^{-1}H^*_{\mathbb{T}}(e^n_\alpha, \partial e^{n-1}_\alpha)$. Since, by Lemma 3.4.2,

$$\epsilon : u^{-1}H^*_{\mathbb{T}}(e^n_\alpha, \partial e^{n-1}_\alpha) \xrightarrow{\cong} h^*_{\mathbb{T}}(e^n_\alpha, \partial e^{n-1}_\alpha)$$

we obtain

$$\epsilon : u^{-1}H^*_{\mathbb{T}}(X^n, X^{n-1}) \xrightarrow{\cong} h^*_{\mathbb{T}}(X^n, X^{n-1})$$

for every $n \geq 0$.

Let us show that the natural homomorphism ϵ commutes also with the boundary homomorphism of the long exact sequences of (X^n, X^{n-1}, A). Let $E^{-1} \subset X^n$ and $E^{n-1} \subset X^n$ be neighbourhoods of A and X^{n-1}, respectively, such that the inclusions

$$(X^n, A) \longrightarrow (X^n, E^{-1}) \quad \text{and} \quad (X^n, X^{n-1}) \longrightarrow (X^n, E^{n-1})$$

are homotopy equivalences. Then the commutative diagram

$$\begin{array}{ccccc}
u^{-1}H^*_{\mathbb{T}}(X^n, X^{n-1}) & \longrightarrow & u^{-1}H^*_{\mathbb{T}}(X^n, A) & \longrightarrow & u^{-1}H^*_{\mathbb{T}}(X^{n-1}, A) \\
\epsilon \downarrow & & \epsilon \downarrow & & \epsilon \downarrow \\
h^*_{\mathbb{T}}(X^n, X^{n-1}) & \longrightarrow & h^*_{\mathbb{T}}(X^n, A) & \longrightarrow & h^*_{\mathbb{T}}(X^{n-1}, A)
\end{array}$$

is induced by the homomorphism of short exact sequences

$$\begin{array}{ccccc}
S^*(X^n, E^{n-1})[u, u^{-1}] & \longrightarrow & S^*(X^n, E^{-1})[u, u^{-1}] & \longrightarrow & S^*(E^{n-1}, E^{-1})[u, u^{-1}] \\
\downarrow & & \downarrow & & \downarrow \\
C^0(X^n; \mathcal{S}^*_{(X^n, X^{n-1})}) & \longrightarrow & C^0(X^n; \mathcal{S}^*_{(X^n, A)}) & \longrightarrow & C^0(X^n; (\mathcal{S}^*_{(X^n, A)})_{X^n|X^{n-1}})
\end{array}$$

since $\mathcal{S}^\bullet_{(X^{n-1},A)} \hookrightarrow (\mathcal{S}^\bullet_{(X^n,A)})_{X^n|X^{n-1}}$ is a quasi-isomorphism of DG sheaves on X^{n-1}. Therefore ϵ commutes also with δ.

By induction on skeleta and 5-lemma applied to the homomorphisms ϵ of the long exact sequences of $u^{-1}H^*_{\mathbb{T}}$ to the long exact sequences of $h^*_{\mathbb{T}}$, we obtain the desired isomorphism. $\boxed{5.2.3}$

This proposition and Milnor's axiom [Milnor 62] for Jones-Petrack cohomology imply the following.

Corollary 5.2.4. *For a pair of* $\mathbb{T}$*-complexes* (X, A) *there exists a short exact sequence*

$$0 \to \varprojlim{}^1 u^{-1}H^*_{\mathbb{T}}(X^p, A)_{-1} \longrightarrow h^*_{\mathbb{T}}(X, A) \longrightarrow \varprojlim u^{-1}H^*_{\mathbb{T}}(X^p, A) \to 0$$

where $(u^{-1}H^*_{\mathbb{T}}(X^p, A)_{-1})^n = (u^{-1}H^*_{\mathbb{T}}(X^p, A))^{n-1}$.

5.3. Cellular equivariant cohomology

In this section we express $\mathbb{T}$-equivariant cohomology of nice $\mathbb{T}$-complexes as the homology of some double complexes.

Definition 5.3.1. Let (X, A) be a pair of $\mathbb{T}$-complexes such that there exists a CW decomposition F of (X, A) with

$$F^n \leq X^n \leq F^{n+1}$$

for every integer $n \geq -1$, where $\leq$ denotes the relation of being a CW-subcomplex and F^n is the n^{th} CW-skeleton of (X, A). Define an operator

$$\mathrm{I} : H^n(F^n, F^{n-1}) \longrightarrow H^{n-1}(F^{n-1}, F^{n-2})$$

as the composite

$$H^n(F^n, F^{n-1}) \to H^n(X^{n-1}, X^{n-2}) \xrightarrow{\mathrm{JB}} H^{n-1}(X^{n-1}, X^{n-2}) \to H^{n-1}(F^{n-1}, F^{n-2}).$$

The operator I is a differential [2] since the restriction map

$$H^n(X^n, X^{n-1}) \longrightarrow H^n(X^{n-1}, X^{n-2})$$

is trivial. Notice also that all homomorphisms in the composite, except for the first one, are injective. By definition and Proposition 5.1.1 the differential map I anticommutes with the usual CW coboundary homomorphism $d : H^n(F^n, F^{n-1}) \to H^{n+1}(F^{n+1}, F^n)$.

[2] This operator should not be confused with the operator I in $S^*(-)[u]$.

Definition 5.3.2. Define a double complex $(\mathfrak{C}^*(X, A)[u]; \delta, u\mathrm{I})$ by

$$\mathfrak{C}^n(X, A) = H^n(F^n, F^{n-1}).$$

We shall gradually prove the following theorem.

Theorem 5.3.3. *There exists a natural isomorphism*

$$\Phi : \mathrm{H}(\mathfrak{C}^*(X, A)[u]) \longrightarrow H^*_{\mathbf{T}}(X, A)$$

for every pair of $\mathbf{T}$*-complexes* (X, A) *with compatible CW and equivariant cellular structures.*

Proposition 5.3.4. *Let* (X, A) *be a pair of* $\mathbf{T}$*-complexes. There exists a unique homomorphism*

$$\Phi : \frac{\mathrm{Ker}(H^n(X^n, X^{n-1}) \to H^{n+1}(X^{n+1}, X^n))}{\mathrm{Im}(H^{n-1}(X^{n-1}, X^{n-2}) \to H^n(X^n, X^{n-1}))} \longrightarrow H^n_{\mathbf{T}}(X)$$

such that

$$\mathrm{J}i^*\Phi(e) = j^*(e)$$

for $e \in \mathrm{Ker}(H^n(X^n, X^{n-1}) \to H^{n+1}(X^{n+1}, X^n))$, $i : X^n \hookrightarrow X$, $j : X^n \hookrightarrow (X^n, X^{n-1})$.

Proof of 5.3.4: Since $H^m_{\mathbf{T}}(X^n, X^{n-1}) = H^m(X^n, X^{n-1}) = 0$ for $m < n$, the long exact sequence gives the following commutative diagram.

$$\begin{array}{ccccc}
H^n_{\mathbf{T}}(X^n, X^{n-1}) & \longrightarrow & H^n_{\mathbf{T}}(X^n) & \longrightarrow & H^{n+1}_{\mathbf{T}}(X^{n+1}, X^n) \\
\mathrm{J}\downarrow\cong & & \mathrm{J}\downarrow & & \mathrm{J}\downarrow\cong \\
H^n(X^n, X^{n-1}) & \longrightarrow & H^n(X^n) & \longrightarrow & H^{n+1}(X^{n+1}, X^n)
\end{array}$$

Given $e \in \mathrm{Ker}(H^n(X^n, X^{n-1}) \to H^{n+1}(X^{n+1}, X^n)) \leq H^n(X^n, X^{n-1})$ we define

$$\Phi(e) = j^*\mathrm{J}^{-1}(e) \in H^n_{\mathbf{T}}(X^n)$$

and $\Phi(e)$ so defined is in the kernel of $\delta : H^n_{\mathbf{T}}(X^n) \to H^{n+1}_{\mathbf{T}}(X^{n+1}, X^n)$. Therefore the exact sequence of the pair (X^{n+1}, X^n) implies that there is a unique lift $\Phi(e) \in H^n_{\mathbf{T}}(X^{n+1})$. The exact sequence of the pair (X^n, X^{n-1}) implies that $\Phi(e)$ is zero if

$$e \in \mathrm{Im}(H^{n-1}(X^{n-1}, X^{n-2}) \to H^n(X^n, X^{n-1})).$$

Since $i^* : H^n_{\mathbf{T}}(X^{n+k+1}) \to H^n_{\mathbf{T}}(X^{n+k})$ is an isomorphism for every $k \geq 1$, we obtain a unique lift $\Phi(e) \in H^n_{\mathbf{T}}(X) = \lim_{\leftarrow} H^n_{\mathbf{T}}(X^r)$. $\boxed{5.3.4}$

Proposition 5.3.5. *Let (X, A) be a relative $\mathbb{T}$-complex. Let elements $e \in H_{\mathbb{T}}^{n-1}(X^{n-1}, A)$ and $f \in H^{n+1}(X^n, X^{n-1})$ be such that*

$$\mathrm{B}f = -\delta e\,.$$

Then there exists exactly one element $\Phi(e, f) \in H_{\mathbb{T}}^{n+1}(X^n, A)$ such that

$$i^*\Phi(e, f) = \mathrm{S}e \quad \text{and} \quad \mathrm{J}\Phi(e, f) = j_n^* f$$

for $j_n : (X^n, A) \hookrightarrow (X^n, X^{n-1})$ and $i : (X^{n-1}, A) \hookrightarrow (X^n, A)$.

If there exists an element $a \in H^n(X^{n-1}, A)$ such that

$$e = \mathrm{B}a \quad \text{and} \quad f = da$$

then $\Phi(e, f) = 0$.

Proof of 5.3.5: Since $i^* : H_{\mathbb{T}}^{n+1}(X^n, A) \to H_{\mathbb{T}}^{n+1}(X^{n-1}, A)$ is injective there can be only one element $\Phi(e, f)$ with the prescribed properties. The diagram

$$\begin{array}{ccccccc}
H_{\mathbb{T}}^{n+1}(X^n, A) & \to & H_{\mathbb{T}}^{n+1}(X^{n-1}, A) & \to & H_{\mathbb{T}}^{n+2}(X^n, X^{n-1}) & \to & H_{\mathbb{T}}^{n+2}(X^n, A) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_{\mathbb{T}}^{n-1}(X^n, A) & \to & H_{\mathbb{T}}^{n-1}(X^{n-1}, A) & \to & H_{\mathbb{T}}^{n}(X^n, X^{n-1}) & \to & H_{\mathbb{T}}^{n}(X^n, A) \\
 & & & & \uparrow & & \uparrow \\
 & & & & H^{n+1}(X^n, X^{n-1}) & \to & H^{n+1}(X^n, A) \\
 & & & & \uparrow & & \uparrow \\
 & & & & H_{\mathbb{T}}^{n+1}(X^n, X^{n-1}) & \to & H_{\mathbb{T}}^{n+1}(X^n, A)
\end{array}$$

shows that there exists a $\Phi(e, f)$ with $i^*\Phi(e, f) = \mathrm{S}e$. Since $H_{\mathbb{T}}^{n+1}(X^n, X^{n-1}) = 0$, the relative version of Lemma 5.1.2 implies that $\mathrm{J}\Phi(e, f) = j_n^* f$.

The second claim follows from the exactness of the SJB exact sequence. $\boxed{5.3.5}$

Proof of 5.3.3: Every cocycle of the form

$$\alpha \in \mathfrak{C}^n = \mathfrak{C}^n(X, A) \quad d\alpha = 0 \quad \mathrm{I}\alpha = 0$$

gives rise to an element in

$$\frac{\mathrm{Ker}(H^n(X^n, X^{n-1}) \to H^{n+1}(X^{n+1}, X^n))}{\mathrm{Im}(H^{n-1}(X^{n-1}, X^{n-2}) \to H^n(X^n, X^{n-1}))}$$

and thus by Proposition 5.3.4 an element $\Phi(\alpha)$ in $H_{\mathbb{T}}^n(X, A)$ by the following commutative diagram.

$$\begin{tikzcd}
H^n(X^n,X^{n-1}) \arrow[rr, tail] & & H^n(F^n,X^{n-1}) \arrow[r] \arrow[d] & H^{n+1}(X^n,F^n) \\
& H^{n-1}(F^{n-1},F^{n-2}) \arrow[r] & H^n(F^n,F^{n-1}) \arrow[r] \arrow[d] & H^{n+1}(F^{n+1},F^n) \arrow[u] \\
H^{n-1}(X^{n-1},X^{n-2}) \arrow[uu] \arrow[r, tail] & H^{n-1}(F^{n-1},X^{n-2}) \arrow[r] \arrow[u] & H^n(X^{n-1},F^{n-1}) \arrow[r] & H^n(X^{n-1},X^{n-2})
\end{tikzcd}$$

Every lift $\overline{\alpha} \in H^{n-1}(F^{n-1},X^{n-2})$ of $i^*\alpha \in H^n(X^{n-1},F^{n-1})$ uniquely determines an element $\alpha - d\overline{\alpha}$ in $H^n(X^n,X^{n-1})$. This element is mapped to zero in $H^{n+1}(F^{n+1},F^n)$ and thus to zero in $H^{n+1}(X^{n+1},X^n)$. The lift $\overline{\alpha}$ is determined up to an element of $H^{n-1}(X^{n-1},X^{n-2})$. By Proposition 5.3.4 $\alpha - d\overline{\alpha}$ gives rise to an element $\Phi(\alpha) \in H^n_{\mathbb{T}}(X,A)$. We can then extend Φ linearly in u to all cocycles of the form $u^i\alpha$ with $d\alpha = 0$ and $\mathrm{I}\alpha = 0$.

Notice that if $\alpha = d\beta$ and $\mathrm{I}\beta = 0$ then $\Phi(\alpha) = 0$. If $u\alpha = \mathrm{I}\gamma$ for some γ such that $d\gamma = 0$, then $\mathrm{JB}\gamma \in H^n(X^n,X^{n-1})$ lifts α. Since $\mathrm{SB}\gamma = 0$ we obtain $\Phi(u\alpha) = 0$.

For a cocycle of the form

$$\alpha + u\beta \quad \text{such that} \quad d\alpha = 0, \quad \mathrm{I}\beta = 0, \quad \mathrm{I}\alpha = -d\beta$$

for $\alpha \in H^n(F^n,F^{n-1})$ and $\beta \in H^{n-2}(F^{n-2},F^{n-3})$ observe the diagram

$$\begin{tikzcd}
& & & H^{n-1}(F^{n-1},X^{n-2}) \arrow[d] \\
& & & H^{n-1}(F^{n-1},F^{n-2}) \arrow[d] \\
H^{n-2}(X^{n-2},X^{n-3}) \arrow[rr, tail] & & H^{n-2}(F^{n-2},X^{n-3}) \arrow[r] \arrow[d] \arrow[ur] & H^{n-1}(X^{n-2},F^{n-2}) \\
& H^{n-3}(F^{n-3},F^{n-4}) \arrow[r] & H^{n-2}(F^{n-2},F^{n-3}) \arrow[d] & \\
H^{n-3}(X^{n-3},X^{n-4}) \arrow[uu] \arrow[r, tail] & H^{n-3}(F^{n-3},X^{n-4}) \arrow[r] \arrow[u] & H^{n-2}(X^{n-3},F^{n-3}) \arrow[r] & H^{n-2}(X^{n-3},X^{n-4})
\end{tikzcd}$$

For $\beta \in H^{n-2}(F^{n-2},F^{n-3})$ there exists a lift $\overline{\beta} \in H^{n-3}(F^{n-3},X^{n-4})$, determined up to $H^{n-3}(X^{n-3},X^{n-4})$, and we can lift $\beta - d\overline{\beta}$ uniquely to $H^{n-2}(F^{n-2},X^{n-3})$. Since

$$d\beta = -\mathrm{I}\alpha \in H^{n-1}(F^{n-1},F^{n-2}),$$

we have

$$d(\beta - d\overline{\beta}) = 0 \in H^{n-1}(X^{n-2},F^{n-2})$$

and so obtain a lift $J\tilde{\beta} \in H^{n-2}(X^{n-2}, X^{n-3})$ of $\beta - d\tilde{\beta}$, determined up to $\mathrm{Im}(H^{n-3}(X^{n-3}, X^{n-4}) \to H^{n-2}(X^{n-2}, X^{n-3}))$. Thus we have elements $\alpha' \in H^n(X^{n-1}, X^{n-2})$ and $i^*\tilde{\beta} \in H_{\mathbb{T}}^{n-2}(X^{n-2}, A)$ such that $di^*\tilde{\beta} = -\mathrm{B}\alpha'$. By Proposition 5.3.5 there exists an element $\Phi(\alpha, \beta) \in H_{\mathbb{T}}^n(X^{n-1}, A)$ such that

$$i^*\Phi(\alpha,\beta) = \mathrm{S}i^*\tilde{\beta} \quad \text{and} \quad \mathrm{J}\Phi(\alpha,\beta) = j_n^*\alpha'.$$

Since $d\alpha \in \mathrm{ImI}$ we can lift $\mathrm{J}\Phi(\alpha,\beta) \in H^n(X^{n-1}, A)$ uniquely via $j^*\alpha \in H^n(F^n, A)$ to $\hat{\Phi} \in H^n(X^n, A)$ by the following commutative diagram. For simplicity we write the diagrams for the absolute case, ($A = \emptyset$), only.

$$\begin{tikzcd}
 & & & H^{n+1}(F^{n+1}, X^n) \arrow[d] \\
H^n(X^{n+1}) \arrow[d] & & H^n(F^n, F^{n-1}) \arrow[r] \arrow[d] & H^{n+1}(F^{n+1}, F^n) \arrow[d] \\
H^n(X^n) \arrow[r, mapsto] \arrow[d] & & H^n(F^n) \arrow[r] \arrow[ur] \arrow[dr] & H^{n+1}(X^n, F^n) \\
H^{n+1}(X^{n+1}, X^n) & & & H^n(X^{n-1})
\end{tikzcd}$$

Since $d\alpha$ is in fact zero and since the diagram

$$\begin{tikzcd}
H^n(X^n) \arrow[r] \arrow[d] & H^n(F^n) \arrow[d] \\
H^{n+1}(X^{n+1}, X^n) \arrow[r] & H^{n+1}(F^{n+1}, F^n)
\end{tikzcd}$$

commutes, we obtain $d\hat{\Phi} = 0$ in $H^{n+1}(X^{n+1}, X^n)$ and we can lift $\mathrm{J}\Phi(\alpha,\beta)$ to $H^n(X^{n+1}, A)$. This lift is unique.

The following commutative diagram shows that there exists also a unique lift of $\Phi(\alpha,\beta)$ to $H_{\mathbb{T}}^n(X^n, A)$ such that $\mathrm{J}\Phi = \hat{\Phi} \in H^n(X^n)$ provided that $d\alpha \in \mathrm{ImI}$. The same diagram shows that there exists also a unique lift of $\Phi(\alpha,\beta)$ to $H_{\mathbb{T}}^n(X^{n+1})$ if $d\alpha = 0$.

$$\begin{tikzcd}
 & & H_{\mathbb{T}}^{n+1}(X^{n+1}, X^n) \arrow[dd, "\cong"] & & \\
H_{\mathbb{T}}^n(X^n, X^{n-1}) \arrow[r] \arrow[ddd, "\cong"] & H_{\mathbb{T}}^n(X^n) \arrow[rr] \arrow[ur] \arrow[ddd] & & H_{\mathbb{T}}^n(X^{n-1}) \arrow[r, dash] \arrow[ddd] & H_{\mathbb{T}}^{n+1}(X^n, X^{n-1}) = 0 \arrow[ddd, dash] \\
 & & H^{n+1}(X^{n+1}, X^n) & & \\
 & & & & \\
H^n(X^n, X^{n-1}) \arrow[r] & H^n(X^n) \arrow[rr] \arrow[uur] & & H^n(X^{n-1}) \arrow[r, dash] & H^{n+1}(X^n, X^{n-1})
\end{tikzcd}$$

Notice that if the cocycle $\alpha + u\beta$ is of the form

$$\alpha = da \quad \beta = \mathrm{I}a$$

for some $a \in H^{n-1}(F^{n-1}, F^{n-2})$, we obtain $\Phi(\alpha, \beta) = 0$. This follows from the fact that there is an element $\overline{a} \in H^{n-1}(X^{n-2}, A)$ such that $\Phi(\alpha, \beta)$ is obtained via Proposition 5.3.5 from elements $da \in H^{n}(X^{n-1}, A)$ and $\mathrm{B}a \in H_{\mathbb{T}}^{n-2}(X^{n-2}, A)$.

Let $\alpha_0 + u\alpha_1 + \ldots + u^m\alpha_m$ be a cocycle of degree $n + 2m$, i.e.

$$d\alpha_0 = 0, \quad \mathrm{I}\alpha_i = -d\alpha_{i+1} \text{ for } 0 \leq i \leq m-1 \quad \mathrm{I}\alpha_m = 0.$$

By the previous step $\alpha_{m-1} + u\alpha_m$ determines an element $\Phi(\alpha_{m-1}, \alpha_m) \in H^{n+2}(X^{n+1}, A)$ such that

$$i^*\Phi(\alpha_{m-1}, \alpha_m) + \mathrm{S}i^*\tilde{\alpha}_m \quad \text{and} \quad \mathrm{J}\Phi(\alpha_{m-1}, \alpha_m) = j^*\alpha'_{m-1}$$

and there exists a lift $\dot{\Phi} \in H_{\mathbb{T}}^{n+2}(X^{n+2}, A)$ of $\Phi(\alpha_{m-1}, \alpha_m)$ such that $\mathrm{J}\dot{\Phi}$ restricts to $j^*\alpha_{m-1} \in H^{n+2}(F^{n+2}, A)$. Since the restriction

$$H^{n+3}(X^{n+3}, X^{n+2}) \to H^{n+3}(F^{n+3}, F^{n+2})$$

is injective, the commutative diagram

$$\begin{array}{ccccc}
 & & H_{\mathbb{T}}^{n+2}(X^{n+2}) & \longrightarrow & H_{\mathbb{T}}^{n+3}(X^{n+3}, X^{n+2}) \\
 & & \downarrow & & \downarrow \\
 & & H^{n+2}(X^{n+2}) & \longrightarrow & H^{n+3}(X^{n+3}, X^{n+2}) \\
 & & \downarrow & & \downarrow \\
H^{n+2}(F^{n+2}, F^{n+1}) & \longrightarrow & H^{n+2}(F^{n+2}) & \longrightarrow & H^{n+3}(F^{n+3}, F^{n+2})
\end{array}$$

shows that

$$d\dot{\Phi} = \mathrm{B}\alpha'_{m-2}$$

for the image $\alpha'_{m-2} \in H^{n+4}(X^{n+3}, X^{n+2})$ of $\alpha_{m-2} \in H^{n+4}(F^{n+4}, F^{n+3})$. Thus we obtain by Proposition 5.3.5 an element

$$\Phi(\alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in H_{\mathbb{T}}^{n+4}(X^{n+3}, A).$$

As before $d\alpha_{m-2} \in \mathrm{ImI}$ implies that we obtain a lift

$$\dot{\Phi}(\alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in H_{\mathbb{T}}^{n+4}(X^{n+4}, A)$$

such that $\mathrm{J}\Phi(\alpha_{m-2}, \alpha_{m-1}, \alpha_m)$ restricts to $j^*\alpha_{m-2} \in H^{n+4}(F^{n+4}, F^{n+3})$. We continue in this way and eventually obtain

$$\Phi(\alpha_1, \ldots, \alpha_m) \in H_{\mathbb{T}}^{n+2m}(X^{n+2m-1}, A).$$

and let the quotient map be denoted by J.

Therefore we have the following diagram in dimension $n-1$, where the vertical homomorphisms are the connecting homomorphisms δ.

$$\begin{tikzcd}
 & H^{n-1}(X^{n-1}) \arrow[dd] & \\
\mathrm{H}(\mathfrak{C}^*(X^{n-1})[u]) \arrow[ur, "\underline{\mathrm{J}}"] \arrow[rr, dash] \arrow[dd] & & H_T^{n-1}(X^{n-1}) \arrow[ul, "\mathrm{J}"'] \arrow[dd] \\
 & H^n(X^n, X^{n-1}) & \\
\mathrm{H}(\mathfrak{C}^*(X^n, X^{n-1})[u]) \arrow[ur] \arrow[rr] & & H_T^{n-1}(X^n, X^{n-1}) \arrow[ul, "\cong"']
\end{tikzcd}$$

The vertical parallelogram-shaped subdiagrams obviously commute. By construction also the triangular subdiagrams commute since

$$\mathrm{J}\Phi(\alpha_0, \ldots, \alpha_m) \in H_T^{n-1}(X^{n-1}, A),$$

for a cocycle

$$\alpha_0 + u.\alpha_1 + \ldots + u^m.\alpha_m,$$

is the preimage of $j^*\alpha_0$ by the monomorphism

$$H^{n-1}(X^{n-1}, A) \to H^{n-1}(F^{n-1}, A).$$

Therefore the subdiagram involving the two connecting homomophisms commutes as well. Thus Φ induces a morphism of long exact sequences of triples (X^n, X^{n-1}, A).

Let us prove that Φ is an isomorphism on (X^n, X^{n-1}). In dimension n we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{H}^n(\mathfrak{C}^*(X^n, X^{n-1})[u]) \arrow[rr, "\Phi"] \arrow[dr, "\underline{\mathrm{J}}"] & & H_T^n(X^n, X^{n-1}) \arrow[dl, "\mathrm{J}"'] \\
 & H^n(X^n, X^{n-1}) &
\end{tikzcd}$$

The homomorphism J is an isomorphism. The same is true for $\underline{\mathrm{J}}$ as can be seen from the double complex $\mathfrak{C}^*(X^n, X^{n-1})$.

Since

$$\begin{aligned} \mathrm{Ker I} &= \mathrm{Ker}(H^{n+1}(X^n, F^n) \to H^{n+1}(X^n, X^{n-1}) \\ &= \mathrm{Im}(d : H^n(F^n, X^{n-1}) \to H^{n+1}(X^n, F^n)) \end{aligned}$$

we obtain

$$\mathbf{H}^{n+1}(\mathfrak{C}^*(X^n, X^{n-1})[u]) = 0$$

and similarly in dimension $n + 1 + 2k$.

For $\mathbf{H}^{n+2}(\mathfrak{C}^*(X^n, X^{n-1})[u])$ observe that Φ is a $\mathbf{Z}[u]$-module homomorphism and is therefore onto in dimension $n + 2$. Since

$$\mathfrak{C}^*(X^n, X^{n-1})[u] \cong \prod_\alpha \mathfrak{C}^*(e^n_\alpha, \partial e^n_\alpha)[u]$$

the same is true for its homology. Let us calculate the homology of degree $n + 2k$ of the double complex on individual cells.

On fixed-point cells the operator $\underline{S}$ is an isomorphism, therefore Φ is an isomorphism on such a cell in dimension $n + 2k$.

On cells with free action we have

$$\mathrm{Im}(\mathrm{I} : H^{n+1}(X^n, F^n) \to H^n(F^n, X^{n-1})) = \mathrm{Ker}(d : H^n(F^n, X^{n-1}) \to H^{n+1}(X^n, F^n)$$

and so the homology of degree $n + 2$ of the double complex is trivial and so is equivariant cohomology of such a cell. The same is true in every degree of the form $n + 2k$.

On cells of type $\mathbf{Z}/p$, for any $p \in \mathbf{N}$, the exact sequence

$$H^n(e^n, \partial e^n) \rightarrowtail H^n(F^n, \partial e^n) \xrightarrow{d} H^{n+1}(e^n, F^n)$$

shows that $\mathrm{Ker}d/\mathrm{Im I} \cong \mathbf{Z}/p$ in degree $n + 2k$. As the homomorphism Φ is in this case an epimorphism between two isomorphic finite groups, it is an isomorphism.

Since we have shown that Φ is an isomorphism on every n-cell, it is also an isomorphism on their product and thus on the pair (X^n, X^{n-1}).

We have obtained

$$\Phi : \mathbf{H}(\mathfrak{C}^*(X^0, A)[u]) \xrightarrow{\cong} H^*_{\mathbb{T}}(X^0, A)$$

and by 5-lemma applied to the long exact sequence of (X^1, X^0, A) we obtain

$$\Phi : \mathbf{H}(\mathfrak{C}^*(X^1, A)[u]) \xrightarrow{\cong} H^*_{\mathbb{T}}(X^1, A).$$

By induction we obtain

$$\Phi : \mathbf{H}(\mathfrak{C}^*(X^n, A)[u]) \xrightarrow{\cong} H^*_{\mathbb{T}}(X^n, A)$$

for any $n \in \mathbf{N}$ and since $\lim^1$ functor is zero in both cases, we obtain

$$\Phi : \mathbf{H}(\mathfrak{C}^*(X, A)[u]) \xrightarrow{\cong} H^*_{\mathbb{T}}(X, A).$$

$\boxed{5.3.3}$

Since Φ is an isomorphism of $\mathbb{Z}[u]$-modules we obtain

Corollary 5.3.6. *There is an isomorphism*

$$\Phi : \mathbb{H}(\mathbb{C}^*(X, A)[u, u^{-1}]) \to u^{-1} H^*_{\mathbb{T}}(X, A).$$

Appendix A

More about Čech cohomology

Here we discuss Čech cohomology of DG presheaves in detail. We prove what we promised in Section 3.2. Then we show that there is a natural isomorphism

$$\check{H}^{*}(X;\mathcal{P}^{\bullet}) \cong H^{*}(X;\mathcal{S}^{\bullet})$$

for every DG presheaf $\mathcal{P}^{\bullet}$ on a compact space X, where $\mathcal{S}^{\bullet}$ is the DG sheaf associated to $\mathcal{P}^{\bullet}$.

This is an essentially straightforward generalisation of the proof of the analogous statement for nongraded presheaves and sheaves in [Godement 58].

A.1. Definition

Let us first repeat the definiton of Čech cohomology. Let X be a topological space, let $\mathfrak{U} = \{U_j; j \in J\}$ be an open covering of X and

$$U_{j_0 \ldots j_p} = \bigcap_{i=0,\ldots,p} U_{j_i}$$

for $j_i \in J$. Let $\mathcal{P}^{\bullet}$ be a differential graded presheaf of R-modules on X. Then we can construct a double complex $C^{*}(\mathfrak{U};\mathcal{P}^{\bullet})$ of Čech cochains with coefficients in the differential graded presheaf $\mathcal{P}^{\bullet}$ by:

$$C^{p}(\mathfrak{U};\mathcal{P}^{q}) = \prod_{(i_0,\ldots,i_p) \in J^{p+1}} \mathcal{P}^{q}(U_{i_0 \ldots i_p}).$$

For $\gamma \in C^{p}(\mathfrak{U};\mathcal{P}^{q})$ let $\gamma(U_{i_0 \ldots i_p}) \in \mathcal{P}^{q}(U_{i_0 \ldots i_p})$ be its $U_{i_0 \ldots i_p}$-component. The differentials are

$$\delta : C^{p}(\mathfrak{U};\mathcal{P}^{q}) \to C^{p+1}(\mathfrak{U};\mathcal{P}^{q})$$

defined by

$$(\delta\gamma)(U_{i_0 \ldots i_{p+1}}) = \sum_{j=0}^{p+1} (-1)^{j} \gamma(U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}) |_{U_{i_0 \ldots i_{p+1}}}$$

for $\gamma \in C^p(\mathfrak{U};\mathcal{P}^q)$ and

$$d : C^p(\mathfrak{U};\mathcal{P}^q) \to C^p(\mathfrak{U};\mathcal{P}^{q+1})$$

determined by $d : \mathcal{P}^\bullet \to \mathcal{P}^{\bullet+1}$. Cech cohomology of $\mathfrak{U}$ and $\mathcal{P}^\bullet$ is defined as

$$\check{H}^*(\mathfrak{U};\mathcal{P}^\bullet) = \mathrm{H}(\mathrm{STot}(C^*(\mathfrak{U};\mathcal{P}^\bullet)))\,.$$

If $k : \mathcal{B}^\bullet \to \mathcal{A}^\bullet$ is a DG presheaf homomorphism over f, where $\mathcal{A}^\bullet$ is a DG presheaf on a space X, $\mathcal{B}^\bullet$ a DG presheaf on a space Y and $f : X \to Y$ a map, it induces a morphism of double complexes

$$f^{*,\bullet} : C^*(\mathfrak{U};\mathcal{B}^\bullet) \to C^*(f^{-1}\mathfrak{U};\mathcal{A}^\bullet)\,.$$

Let $\mathfrak{V}$ be a refinement of the covering $\mathfrak{U}$. In this case we can choose a function, called projection, $\pi : \mathfrak{V} \to \mathfrak{U}$ such that $V \subset \pi(V)$ for every $V \in \mathfrak{V}$. Such a projection induces a morphism of double complexes

$$\pi : C^*(\mathfrak{U};\mathcal{P}^\bullet) \to C^*(\mathfrak{V};\mathcal{P}^\bullet)$$

defined by

$$(\pi\gamma)(V_{i_0\ldots i_p}) = \gamma(\pi(V_{i_0}) \cap \ldots \cap \pi(V_{i_p}))\,|_{V_{i_0\ldots i_p}}$$

for $\gamma \in C^p(\mathfrak{U};\mathcal{P}^q)$ and $V_{i_j} \in \mathfrak{V}$. Let us check that the homomorphism π commutes with the differentials in the two double complexes: For simplicity let σ denote a simplex $V_{i_0\ldots i_{p+1}}$, let $\pi(\sigma)$ denote the intersection $\pi(V_{i_0}) \cap \ldots \cap \pi(V_{i_{p+1}})$, let σ_j denote $V_{i_0\ldots\hat{i}_j\ldots i_{p+1}}$ and let $\pi(\sigma)_j = \pi(\sigma_j) = \pi(V_{i_0}) \cap \ldots \cap \pi(\hat{V}_{i_j}) \cap \ldots \cap \pi(V_{i_{p+1}})$. Then

$$\begin{aligned}
(\pi\delta\gamma)(\sigma) &= \delta\gamma(\pi(\sigma))\,|_\sigma \\
&= \left(\textstyle\sum_{j=0}^{p+1}(-1)^j\gamma(\pi(\sigma)_j)\right)|_\sigma \\
&= \textstyle\sum_{j=0}^{p+1}(-1)^j\gamma(\pi(\sigma)_j)\,|_\sigma
\end{aligned}$$

and

$$\begin{aligned}
(\delta\pi\gamma)(\sigma) &= \textstyle\sum_{j=0}^{p+1}(-1)^j(\pi\gamma)(\sigma_j)\,|_\sigma \\
&= \textstyle\sum_{j=0}^{p+1}(-1)^j(\gamma(\pi(\sigma_j))\,|_{\sigma_j})\,|_\sigma \\
&= \textstyle\sum_{j=0}^{p+1}(-1)^j\gamma(\pi(\sigma_j))\,|_\sigma\,.
\end{aligned}$$

As $\pi(\sigma)_j = \pi(\sigma_j)$ these two sums coincide and so π commutes with the differentials δ. The fact that π commutes also with the differential maps d follows directly from the definition of the differential presheaf (i.e. the differential commutes with the restrictions).

Lemma A.1.1. *Two projections $\pi_1, \pi_2 : \mathfrak{V} \to \mathfrak{U}$ induce cochain homotopic morphisms*

$$\mathrm{STot}(C^{\bullet}(\mathfrak{U}; \mathcal{P}^{\bullet})) \to \mathrm{STot}(C^{\bullet}(\mathfrak{V}; \mathcal{P}^{\bullet})) .$$

Proof of A.1.1: For a nonempty intersection $\sigma = V_{0 \dots p}$ of $p+1$ elements of $\mathfrak{V}$ and an integer j such that $0 \leq j \leq p$ let us define the corresponding intersection $\tilde{\sigma}_j$ of $p+2$ elements of $\mathfrak{U}$ by

$$\tilde{\sigma}_j = \pi_1(V_0) \cap \dots \cap \pi_1(V_j) \cap \pi_2(V_j) \cap \dots \cap \pi_2(V_p) .$$

Let $h^p : C^{p+1}(\mathfrak{U}; \mathcal{P}^q) \to C^p(\mathfrak{V}; \mathcal{P}^q)$ be the homotopy operator defined by

$$(h^p \gamma)(\sigma) = \sum_{j=0}^{p} (-1)^j \gamma(\tilde{\sigma}_j) \restriction_{\sigma} .$$

We have

$$\begin{aligned} h(\delta + (-1)^p d) + (\delta + (-1)^{p-1} d) h &= h\delta + \delta h + (-1)^p h d + (-1)^{p-1} d h \\ &= h\delta + \delta h \end{aligned}$$

as d and h commute. A straightforward calculation shows that $h\delta + \delta h = \pi_2 - \pi_1$ and thus π_1 and π_2 are indeed cochain homotopic maps. $\boxed{\text{A.1.1}}$

Therefore for every refinement $\mathfrak{V}$ of $\mathfrak{U}$ there is a well-defined homomorphism in cohomology, independent of the projections:

$$\check{H}^{\bullet}(\mathfrak{U}; \mathcal{P}^{\bullet}) \to \check{H}^{\bullet}(\mathfrak{V}; \mathcal{P}^{\bullet}) .$$

The open coverings of a space form a direct system with respect to the relation of refining, therefore we can define

$$H^{\bullet}(X; \mathcal{P}^{\bullet}) = \varinjlim \check{H}^{\bullet}(\mathfrak{U}; \mathcal{P}^{\bullet}) .$$

Lemma A.1.2. *The inclusion of the subcomplex*

$$\mathrm{STot}(C^{\bullet}(\mathfrak{U}; \mathcal{P}^{\bullet})) \hookrightarrow \mathrm{STot}(C^{\bullet}(\mathfrak{U}; \mathcal{P}^{\bullet}))$$

is a homotopy equivalence, where

$$\check{C}^p(\mathfrak{U}; \mathcal{P}^q) = \{\gamma \in C^p(\mathfrak{U}; \mathcal{P}^q) ; \gamma(U_{i_{\epsilon(0)} \dots i_{\epsilon(p)}}) = \mathrm{sign}(\epsilon) \gamma(U_{i_0 \dots i_p})\}$$

are the alternating cochains (ϵ is a permutation of $\{0, \dots, p\}$).

Proof of A.1.2: For this proof we shall use the fact that $C^*(\mathfrak{U};\mathcal{P})$ is the DG module of cochains on a simplicial complex, called the nerve of $\mathfrak{U}$, with a system of coefficients.

A system $\mathcal{L}$ of coefficients on a simplicial complex K is an assignment

$$s \mapsto \mathcal{L}(s)$$
$$\{s \subset t\} \mapsto \{\mathcal{L}(s) \to \mathcal{L}(t)\}$$

for all simplexes $s, t \in K$, where $\mathcal{L}(s)$ and $\mathcal{L}(t)$ are R-modules and the maps $\mathcal{L}(s) \to \mathcal{L}(t)$, called the restriction homomorphisms, are homomorphisms of R-modules such that

(a) $\{s = t\} \mapsto \{\mathcal{L}(s) \xrightarrow{\mathrm{id}} \mathcal{L}(t)\}$

(b) if $s \subset t \subset u$, then

$$\{s \subset u\} \mapsto \{\mathcal{L}(s) \to \mathcal{L}(t) \to \mathcal{L}(u)\}.$$

A homomorphism $\theta : \mathcal{L} \to \mathcal{M}$ of systems of coefficients on the simplicial complex K is a collection $\theta(s) : \mathcal{L}(s) \to \mathcal{M}(s)$ of homomorphisms of R-modules, for every simplex $s \in K$, such that these homomorphisms commute with the restriction homomorphisms in $\mathcal{L}$ and $\mathcal{M}$, respectively.

If $\mathfrak{U} = (U_i)_{i \in J}$ is an open covering of the space X, we can endow the set J with a structure of a simplicial complex in which every set $\{i_0, \ldots, i_n\}$ such that

$$U_{i_0} \cap \ldots \cap U_{i_n} \neq \emptyset$$

is an n-simplex in J. This simplicial complex is called the nerve of the covering $\mathfrak{U}$.

A presheaf $\mathcal{P}$ on X determines a system of coefficients on the nerve of every open covering $\mathfrak{U} = (U_i)_{i \in J}$ of X by

$$\{i_0, \ldots, i_n\} \mapsto \mathcal{P}(U_{i_0} \cap \ldots \cap U_{i_n})$$

and the restriction homomorphisms of the system of coefficients are the restriction homomorphisms of the presheaf.

For any simplicial complex K and any system of coefficients $\mathcal{L}$ there is a homotopy equivalence $C^*(K;\mathcal{L}) \xrightarrow{\simeq} C^*(K;\mathcal{L})$ which is natural with respect to morphisms of systems of coefficients. This is proved in [Godement 58, pp. 59-62].

In our case the differential maps d in $(\mathcal{P}^\bullet, d)$ are presheaf maps and therefore morphisms of systems of coefficients on the nerve of $\mathfrak{U}$.

Let us denote the inclusion $C^p(\mathfrak{U};\mathcal{P}^q) \hookrightarrow C^p(\mathfrak{U};\mathcal{P}^q)$ by $i_{p,q}$, its homotopy inverse by $\pi_{p,q}$. Let

$$h_{p,q} : C^p(\mathfrak{U};\mathcal{P}^q) \longrightarrow C^{p-1}(\mathfrak{U};\mathcal{P}^q)$$
$$k_{p,q} : C^p(\mathfrak{U};\mathcal{P}^q) \longrightarrow C^{p-1}(\mathfrak{U};\mathcal{P}^q)$$

be the cochain homotopies so that

$$\pi_{p,q}\imath_{p,q} - \mathrm{Id} = \delta k_{p,q} + k_{p+1,q}\delta$$
$$\imath_{p,q}\pi_{p,q} - \mathrm{Id} = \delta h_{p,q} + h_{p+1,q}\delta$$

for every $p, q \in \mathbf{Z}$. Because of naturality we obtain

$$(\delta + (-1)^{p-1}d)k_{p,q} + k_{p+1,q}(\delta + (-1)^p d) = \pi_{p,q}\imath_{p,q}$$
$$(\delta + (-1)^{p-1}d)h_{p,q} + h_{p+1,q}(\delta + (-1)^p d) = \imath_{p,q}\pi_{p,q}.$$

Therefore

$$\sum_{p+q=n} h_{p,q} : (\mathrm{STot}(C^*(\mathfrak{U};\mathcal{P}^\bullet)))^n \longrightarrow (\mathrm{STot}(C^*(\mathfrak{U};\mathcal{P}^\bullet)))^{n-1}$$
$$\sum_{p+q=n} k_{p,q} : (\mathrm{STot}(C^*(\mathfrak{U};\mathcal{P}^\bullet)))^n \longrightarrow (\mathrm{STot}(C^*(\mathfrak{U};\mathcal{P}^\bullet)))^{n-1}$$

are the homotopy operators which make the inclusion

$$\mathrm{STot}(C^*(\mathfrak{U};\mathcal{P}^\bullet)) \longrightarrow \mathrm{STot}(C^*(\mathfrak{U};\mathcal{P}^\bullet))$$

into a homotopy equivalence. $\boxed{\text{A.1.2}}$

Corollary A.1.3.

$$\check{H}^*(X;\mathcal{P}^\bullet) \cong \varinjlim \mathrm{H}(\mathrm{STot}(C^*(\mathfrak{U};\mathcal{P}^\bullet)))$$

as functors on the category of differential presheaves and homomorphisms over maps.

Proof of A.1.3: Projections to refinements map alternating cochains to alternating cochains and homomorphisms of DG presheaves map alternating cochains to alternating cochains. Therefore by Lemma A.1.1 and Lemma A.1.2 we obtain the canonical isomorphism in cohomology. $\boxed{\text{A.1.3}}$

There is another version of Čech cohomology of DG presheaves. Let $\mathfrak{Cov}_X$ be the category of all open coverings of X of the form $\mathfrak{U} = \{U_x; x \in X\}$ such that $x \in U_x$ for every $x \in X$ and projections $r : \mathfrak{V} \to \mathfrak{U}$, for $\mathfrak{U}, \mathfrak{V} \in \mathfrak{Cov}_X$, such that $r(V_x) = U_x$. The category $\mathfrak{Cov}_X$ is a filtered category and we can define

$$\check{C}^*(X;\mathcal{P}^\bullet) = \varinjlim C^*(\mathfrak{U};\mathcal{P}^\bullet)$$

where the direct limit is over all $\mathfrak{U} \in \mathfrak{Cov}_X$. $\check{C}^*(X;\mathcal{P}^\bullet)$ is a double complex. Direct limits commute with the homology functor and with direct sums. Therefore the fact that $\mathfrak{Cov}_X$ is cofinal in the category of all open coverings of X implies that there is a canonical isomorphism

$$\check{H}^*(X;\mathcal{P}^\bullet) \cong \mathrm{H}(\mathrm{STot}(\check{C}^*(X;\mathcal{P}^\bullet))).$$

This isomorphism is natural with respect to homomorphisms over maps:

A map $f : X \to Y$ induces a map

$$\mathfrak{Cov}_Y \to \mathfrak{Cov}_X, \quad \{U_y\} \mapsto \{V_x = f^{-1}(U_{f(x)})\}$$

and if $k : \mathcal{R}^\bullet \to \mathcal{P}^\bullet$ is a DG presheaf homomorphism, it induces a morphism $k : \check{C}^\bullet(Y; \mathcal{R}^\bullet) \to \check{C}^\bullet(X; \mathcal{P}^\bullet)$. As the direct limit commutes with the homology operator naturally with respect to morphisms of direct systems there is a commutative diagram

$$\begin{array}{ccc} \check{H}(Y; \mathcal{R}^\bullet) & \xrightarrow{k} & \check{H}(X; \mathcal{P}^\bullet) \\ \| & & \| \\ \mathrm{H}(\mathrm{STot}(\check{C}^\bullet(Y; \mathcal{R}^\bullet))) & \xrightarrow{k} & \mathrm{H}(\mathrm{STot}(\check{C}^\bullet(X; \mathcal{P}^\bullet))) \end{array}$$

in which $\|$ denotes the natural isomorphisms and k the two homomorphisms induced by the homomorphism k over f.

A.2. Properties

The main property which we show here is that Čech cohomology of a DG presheaf agrees with Čech cohomology of its associated DG sheaf.

Lemma A.2.1. *Let F be a functor from the category of DG presheaves on a space X to an abelian category, with the following properties:*

(1) F is exact.

(2) F is trivial on DG presheaves whose associated DG sheaf is trivial.

Then

(1) F takes the same value on a DG presheaf as on its associated DG sheaf.

(2) F is exact also when regarded, in the natural way, as a functor on the category of DG sheaves on X.

Proof of A.2.1: Let $\mathcal{P}^\bullet$ be a DG presheaf on X. Let $\mathcal{N}^\bullet$ be the DG subpresheaf of $\mathcal{P}^\bullet$ consisting of locally zero elements,i.e. let $\gamma \in \mathcal{P}^n(U)$, then $\gamma \in \mathcal{N}^n(U)$ if and only if for every point $x \in U$ there exists a neighbourhood $V = V(\gamma, x)$ for which $\gamma\,|_V = 0$. Thus we obtain an exact sequence of DG presheaves

$$0 \to \mathcal{N}^\bullet \to \mathcal{P}^\bullet \to \mathcal{Z}^\bullet \to 0$$

where $\mathcal{Z}^\bullet$ is the quotient DG presheaf. Then $\mathcal{Z}^\bullet$ has the property that zero is the only locally zero element. A DG presheaf with this property naturally injects into its associated DG sheaf $\mathcal{S}^\bullet$: Let U be an open subset of X and let $\gamma_1, \gamma_2 \in \mathcal{Z}^n(U)$. Set $\gamma_i(x) = \operatorname{dir\,lim} \gamma\,|_{V(x)}$ for $i = 1, 2$, where the direct limit is taken over open neighbourhoods $V(x)$ of x in U. Then $\gamma_1(x) = \gamma_2(x)$

implies the existence of a neighbourhood $W \subseteq U$ of x such that $\gamma_1|_W = \gamma_2|_W$. If $\gamma_1(x) = \gamma_2(x)$ for every x in U, then $\gamma_1 - \gamma_2$ is locally zero and so by assumption zero. Therefore there is a short exact sequence of DG presheaves

$$0 \to \mathcal{J}^\bullet \to \mathcal{S}^\bullet \to \mathcal{Q}^\bullet \to 0$$

where $\mathcal{Q}^\bullet$ is the quotient DG presheaf. The differential presheaves $\mathcal{N}^\bullet$ and $\mathcal{Q}^\bullet$ give rise to the trivial DG sheaf, therefore we obtain from the above exact sequences a canonical isomorphism

$$F(\mathcal{P}^\bullet) \xrightarrow{\cong} F(\mathcal{J}^\bullet) \xrightarrow{\cong} F(\mathcal{S}^\bullet).$$

Let

$$0 \to \mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{D}^\bullet \to 0$$

be an exact sequence of DG sheaves. Let us define a DG presheaf $\mathcal{D}_0^\bullet$ by

$$\mathcal{D}_0^\bullet(U) = \operatorname{Im}(\mathcal{B}^\bullet(U) \to \mathcal{D}^\bullet(U)).$$

We obtain an exact sequence of DG presheaves

$$0 \to \mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{D}_0^\bullet \to 0$$

and therefore the sequence

$$0 \to F(\mathcal{A}^\bullet) \to F(\mathcal{B}^\bullet) \to F(\mathcal{D}_0^\bullet) \to 0$$

is exact. The associated DG sheaf of $\mathcal{D}_0^\bullet$ is $\mathcal{D}^\bullet$ and so we obtain an exact sequence

$$0 \to F(\mathcal{A}^\bullet) \to F(\mathcal{B}^\bullet) \to F(\mathcal{D}^\bullet) \to 0$$

proving also the second assertion. $\boxed{\text{A.2.1}}$

Corollary A.2.2. *Let $\mathcal{P}^\bullet$ be a DG presheaf on a space X and let $\mathcal{S}^\bullet$ be the DG sheaf induced by $\mathcal{P}^\bullet$. There is an isomorphism*

$$\check{H}^*(X;\mathcal{P}^\bullet) \xrightarrow{\cong} \check{H}^*(X;\mathcal{S}^\bullet)$$

natural with respect to homomorphisms over maps. For every short exact sequence of DG sheaves there is a long exact sequence of $\check{H}^*(X;-)$.

Proof of A.2.2: The proof is very similar to the one of the preceding lemma, let us use therefore the same data and the same notation without repeating the definitions.

The short exact sequences of DG presheaves

$$0 \to \mathcal{N}^\bullet \to \mathcal{P}^\bullet \to \mathcal{J}^\bullet \to 0, \quad 0 \to \mathcal{J}^\bullet \to \mathcal{S}^\bullet \to \mathcal{Q}^\bullet \to 0$$

which are natural with respect to homomorphisms over maps, give rise to the corresponding long exact sequences in Cech cohomology by Proposition 3.2.5. The cohomology of $\mathcal{N}^\bullet$ and $\mathcal{Q}^\bullet$ is trivial. This proves the first claim.

For the second assertion let

$$0 \to \mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{D}^\bullet \to 0$$

be an exact sequence of DG sheaves. Then it induces a short exact sequence of DG presheaves

$$0 \to \mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{D}_0^\bullet \to 0$$

as in the proof of the lemma above. This in turn induces a long exact sequence of $\check{H}^*(X; -)$. By the first part of this proof we can replace $\check{H}^*(X; \mathcal{D}_0^\bullet)$ by $\check{H}^*(X; \mathcal{D}^\bullet)$ in the long exact sequence. $\boxed{\text{A.2.2}}$

A.3. Isomorphism theorem

Theorem A.3.1. *For every DG sheaf $\mathcal{S}^\bullet$ on a compact space X there is an isomorphism*

$$H^*(X; \mathcal{S}^\bullet) \cong \check{H}^*(X; \mathcal{S}^\bullet)$$

and this isomorphism is natural with respect to homomorphisms over maps.

In fact this holds for any metrisable space, but we need it on compact spaces only and the proof is shorter in this case. The proof is a straightforward adaptation of the analogous proof for the (non-differentiable) sheaves [Godement 58, pp. 230-231]. First we need to introduce a new notion:

Resolution induced by a covering

Let $\mathfrak{M} = \{M_i; i \in J\}$ be an open covering of X. Given an open subset U of X we can define an open covering

$$\mathfrak{M} \cap U = \{M_i \cap U; i \in J\}$$

of U with the same indexing set as $\mathfrak{M}$. Let $\mathcal{S}$ be a sheaf on X. In this case we obtain for every open subset U of X a cochain complex

$$C^*(\mathfrak{M} \cap U; \mathcal{S}) = \prod_{(i_0, \dots, i_n) \in J^{n+1}} \mathcal{S}(M_{i_0 \dots i_n} \cap U)$$

with the usual Čech differential. If $V \subseteq U$, there are restriction homomorphisms $\mathcal{S}(\sigma \cap U) \to \mathcal{S}(\sigma \cap V)$ for every non-empty intersection $\sigma = \bigcap_{j=0}^n M_{i_j}$. These induce a cochain map

$$C^*(\mathfrak{M} \cap U; \mathcal{S}) \longrightarrow C^*(\mathfrak{M} \cap V; \mathcal{S}).$$

These cochain maps satisfy the transitivity condition. Therefore for every $n \geq 0$ the assignment

$$U \mapsto C^n(\mathfrak{M} \cap U; \mathcal{S})$$

defines a presheaf on X which we denote by $\mathcal{C}^n(\mathfrak{M}; \mathcal{S})$. $\mathcal{C}^*(\mathfrak{M}; \mathcal{S})$ is a DG presheaf on X. Let us show that it is in fact a DG sheaf. For any subset M of X the assignment $U \mapsto \mathcal{S}(M \cap U)$ defines a presheaf which is a sheaf, let us denote it by $\mathcal{S}_{(M)}$. The presheaf $\mathcal{C}^n(\mathfrak{M}; \mathcal{S})$ is by definition

$$U \mapsto \prod_{\sigma} \mathcal{S}(M_\sigma \cap U)$$

for all ordered $n+1$-tuples σ of J. Therefore it is a direct product of sheaves and thus by [Godement 58, p. 117], itself a sheaf.

If $\mathcal{S}$ is a sheaf on X, $\mathcal{T}$ a sheaf on Y, $f : X \to Y$ a map, then every homomorphism $k : \mathcal{T} \to \mathcal{S}$ over f induces a DG sheaf homomorphism over f

$$k^* : \mathcal{C}^*(\mathfrak{U}; \mathcal{T}) \to \mathcal{C}^*(f^{-1}\mathfrak{U}; \mathcal{S}).$$

From the identities

$$\mathcal{C}^n(\mathfrak{M}; \mathcal{S}) = \prod_{\sigma} \mathcal{S}_{(M_\sigma)}, \quad \mathcal{S}(M_\sigma) = \mathcal{S}_{(M_\sigma)}(X)$$

we obtain the identity

$$C^*(\mathfrak{M}; \mathcal{S}) = (\mathcal{C}^*(\mathfrak{M}; \mathcal{S}))(X).$$

Proof of A.3.1: In the case of a DG sheaf $\mathcal{S}^\bullet$, the differential in $\mathcal{S}^\bullet$ induces a DG sheaf morphism

$$d : \mathcal{C}^*(\mathfrak{M}; \mathcal{S}^\bullet) \longrightarrow \mathcal{C}^*(\mathfrak{M}; \mathcal{S}^{\bullet+1})$$

with $d^2 = 0$ and the differentials commute.

We can do this for every open covering $\mathfrak{U}$ in the family $\mathfrak{Cov}_X$ and define

$$\check{\mathcal{C}}^*(X; \mathcal{S}^\bullet) = \varinjlim \mathcal{C}^*(\mathfrak{U}; \mathcal{S}^\bullet)$$

where the direct limit is taken over the directed family $\mathfrak{Cov}_X$ of open coverings of X. Every DG sheaf homomorphism induces a homomorphism between the corresponding direct limit bidifferential sheaves.

For every grade $\mathcal{S}^\bullet$ the differential sheaf $\check{\mathcal{C}}^*(X; \mathcal{S}^\bullet)$ is a resolution of $\mathcal{S}^\bullet$ and by [Godement 58, pp. 230-231], it is a fine resolution. Therefore we obtain by Theorem 4.2.9 a canonical isomorphism

$$H^n(X; \mathcal{S}^\bullet) \cong \mathrm{H}^n(\mathrm{STot}(\check{\mathcal{C}}^*(X; \mathcal{S}^\bullet)(X))).$$

This isomorphism is natural with respect to DG sheaf homomorphisms over maps because a homomorphism $k : \mathcal{T}^\bullet \to \mathcal{S}^\bullet$ induces a homomorphism of triple complexes

$$C^\bullet(Y; \mathcal{C}^\bullet(Y; \mathcal{T}^\bullet)) \longrightarrow C^\bullet(X; \mathcal{C}^\bullet(X; \mathcal{S}^\bullet)) .$$

When X is compact, there is a canonical isomorphism [Godement 58, p. 162]

$$(\check{\mathcal{C}}^\bullet(X; \mathcal{S}^\bullet))(X) = \varinjlim(\mathcal{C}^\bullet(\mathfrak{U}; \mathcal{S}^\bullet))(X)$$

natural with respect to homomorphisms over maps. Since

$$\varinjlim \mathcal{C}^\bullet(\mathfrak{U}; \mathcal{S}^\bullet)(X) = \varinjlim C^\bullet(\mathfrak{U}; \mathcal{S}^\bullet) = \check{C}^\bullet(X; \mathcal{S}^\bullet)$$

we obtain

$$(\mathcal{C}^\bullet(X; \mathcal{S}^\bullet))(X) = \check{C}^\bullet(X; \mathcal{S}^\bullet)$$

inducing the desired isomorphism. $\boxed{\text{A.3.1}}$

Bibliography

[Atiyah,Bott 84] M. F. Atiyah and R. Bott, The moment map and equivariant cohomology, *Topology*, **23 (1)**,pp. 1-28, 1984.

[Beggs 87] Edwin J. Beggs, The de Rham complex on infinite dimensional manifolds, *Quart. J. Math. Oxford*(2), **38**, pp. 131-154, 1987.

[Bredon 65] Glen E. Bredon, *Sheaf theory*, McGraw-Hill, New York 1965.

[Bredon 72] Glen Eugene Bredon, *Introduction to compact transformation groups*, *Pure and applied mathematics*, Vol. 46, Academic Press, New York, London, 1972.

[Borel 60] Armand Borel, *Seminar on transformation groups*, *Annals of Math. Studies*, Princeton University Press, 1960.

[Goodwillie 85] Thomas G. Goodwillie, Cyclic homology, derivations, and the free loopspace, *Topology*, **24 (2)**, pp. 187-215, 1985.

[Godement 58] Roger Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris 1958.

[Griffiths, Harris 78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, John Wiley & Sons, New York, 1978.

[Hsiang 75] Wu Yi Hsiang,*Cohomology theory of topological transformation groups*,*Ergebnisse der Mathematik und ihrer Grenzgebiete*, Bd. 85, Springer Verlag, 1975.

[Jones 87] John D.S. Jones, Cyclic homology and equivariant homology, *Inventiones mathematicae*, **87**, pp. 403-423, 1987.

[Jones, Petrack 88] J.D.S. Jones and S.B. Petrack, Le théorème des points fixes en cohomologie équivariante en dimension infinie, *C.R. Acad. Sci. Paris Ser. I*, **306**, pp. 75-78, 1988.

[Jones, Petrack 90] J.D.S. Jones and S.B. Petrack, The fixed point theorem in equivariant cohomology, *Trans. Am. Math. Soc.*, **322 (1)**, pp. 35-50, 1990.

[Milnor 62] John W. Milnor, On axiomatic homology theory, *Pacific J. Math.*, **12**, pp. 337-341, 1962.

[Quillen 71] Daniel Quillen, The spectrum of an equivariant cohomology:I, II, *Annals of Math.*, **94**, pp. 549-602, 1971.

[Whitehead 78] George W. Whitehead, *Elements of homotopy theory, Graduate texts in mathematics*, Vol. 61, Springer Verlag, New York,1978.

[Willson 75] Stephen J. Willson, Equivariant homology theories on G-complexes, *Trans. Am. Math. Soc.*, **212**, pp. 155-171, 1975.

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